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SOME RECURRENCE RELATIONS ASSOCIATED WITH THE ALAVI SEQUENCE

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Abstract

This paper considers a modification of the Fibonacci sequence which results in the third order Alavi sequence. Not only are the initial terms quite general but the rule of formation is also modified. Some results are proved to illustrate the underlying structure of the sequence and its relation to known results in the literature. The paper concludes with a suggestion for further research with an arbitrary order extension.

1. Introduction

There have been many generalizations of the Fibonacci sequence. Among them are

- generalizing the initial terms [4];
- generalizing the recurrence relation [5];
- extending the recurrence relation to third order [3];
- extending the recurrence relation to arbitrary order [7];
- coupling the recurrence relations [2].

In this paper we modify the pattern with a third order sequence, from which emerges the sequence attributed by Sloane and Plouffe [9] to Alavi [1]. The first 24 members of this sequence $\{A_n\}_{n\geq 0}$ have the form:

$$\begin{array}{cccc}
a & 2a+b+c & 6a+5b+5c & 22a+21b+21c \\
b & a+2b+c & 5a+6b+5c & 21a+22b+21c \\
c & a+b+2c & 5a+5b+6c & 21a+21b+22c \\
a+b & 3a+3b+2c & 11a+11b+10c & 43a+43b+42c \\
a+c & 3a+2b+3c & 11a+10b+11c & 43a+42b+43c \\
b+c & 2a+3b+3c & 10a+11b+11c & 42a+43b+43c
\end{array} \tag{1.1}$$

where a,b,c are given constants. The coefficients of the members are

1 0 0	2 1 1	6 5 5	22 21 21
0 1 0	1 2 1	5 6 5	21 22 21
0 0 1	1 1 2	5 5 6	21 21 22
1 1 0	3 3 2	11 11 10	43 43 42
1 0 1	3 2 3	11 10 11	43 42 43
0 1 1	2 3 3	10 11 11	42 43 43

It is not immediately obvious that this is indeed a generalization of the Fibonacci sequence though the pattern among the coefficients gives a clue. The corresponding second order case is easier to see:

1 0	1 2	a	a+2b
0 1	3 2	b	3a+2b
1 1	2 3	a+b	2a+3b
2 1	5 3	2a+b	5a+3b

which when a=b=1, becomes {1,1,2,3,3,5,5,8,8,13,13,21,21,…}. We shall outline properties of $\{A_n\}$in relation to some known results and indicate how it could be generalized.

2. Definitions

Let the Alavi sequence have the form

$$\{A_n(a,b,c)\} \equiv \{A_n\} = \{\alpha_n a + \beta_n b + \gamma_n c\}_{n\geq 1}. \tag{2.1}$$

We shal find a formula for these coefficients. Note that

$$\alpha_n + \beta_n + \gamma_n = 2\lfloor \tfrac{1}{3} n \rfloor. \tag{2.2}$$

First, we see that

$$\alpha_1 = 1, \alpha_2 = 2\alpha_2 - 1, \alpha_3 = 2\alpha_2, \alpha_4 = 2\alpha_3 - 1,$$

and, more generally, for each natural number $n \geq 1$:

$$\alpha_1 = 1, \quad \alpha_{2n} = 2\alpha_{2n-1} - 1, \quad \alpha_{2n+1} = 2\alpha_{2n}.$$

These results can be proved directly by induction. Similarly, it can be proved that

$$\begin{array}{lll}
\alpha_1 = 1, & \beta_1 = \alpha_1 - 1, & \gamma_1 = \alpha_1 - 1, \\
\alpha_2 = \alpha_1 - 1, & \beta_2 = \alpha_1, & \gamma_2 = \alpha_1 - 1, \\
\alpha_3 = \alpha_1 - 1, & \beta_3 = \alpha_1 - 1, & \gamma_3 = \alpha_1, \\
\ldots & \ldots & \ldots \\
\alpha_{6k+4} = 2\alpha_{6k+4} - 1, & \beta_{6k+4} = \alpha_{6k+4}, & \gamma_{6k+4} = \alpha_{6k+4} - 1, \\
\alpha_{6k+5} = \alpha_{6k+4}, & \beta_{6k+5} = \alpha_{6k+4} - 1, & \gamma_{6k+5} = \alpha_{6k+4}, \\
\alpha_{6k+6} = \alpha_{6k+4} - 1, & \beta_{6k+6} = \alpha_{6k+4}, & \gamma_{6k+6} = \alpha_{6k+4}, \\
\alpha_{6k+7} = 2\alpha_{6k+7}, & \beta_{6k+7} = \alpha_{6k+7} - 1, & \gamma_{6k+7} = \alpha_{6k+7} - 1, \\
\alpha_{6k+8} = \alpha_{6k+7} - 1, & \beta_{6k+8} = \alpha_{6k+7}, & \gamma_{6k+8} = \alpha_{6k+7} - 1, \\
\alpha_{6k+9} = \alpha_{6k+7} - 1, & \beta_{6k+9} = \alpha_{6k+7} - 1, & \gamma_{6k+9} = \alpha_{6k+7}.
\end{array}$$

Therefore, we must find an explicit form for the sequence

$$\{A_m\}_{m\geq 0} = \{\alpha_{3k+1}\}_{k\geq 0}$$

that has its initial members:

$$1,1,2,3,6,11,22,43,\ldots.$$

From above we can write

$$A_0 = 1, A_{2m+1} = 2A_{2m}, A_{2m+2} = 2A_{2m+1} - 1.$$

3. Main Result

Now we shall prove that for each natural number $m \geq 0$:

$$A_m = 2^m - \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \lfloor 2^{m-1-2i} \rfloor, \tag{3.1}$$

which is equivalent to

$$A_m = 2^m - \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} 2^{m-1-2i} - \lfloor 2^{m-1-2\lfloor \frac{m}{2} \rfloor} \rfloor, \tag{3.2}$$

which is obvious when m=0:

$$A_0 = 2^0.$$

Let us assume that for some m, A_m satisfies (3.2). If m=2k+1, then

$$\begin{aligned} A_{m+1} &= A_{2k+2} \\ &= 2A_{2k+1} - 1 \\ &= 2\left(2^{2k+1} - \sum_{i=0}^{\lfloor \frac{2k+1}{2} \rfloor - 1} 2^{(2k+1)-1-2i} - 2^{(2k+1)-1-2\lfloor \frac{2k+1}{2} \rfloor} \right) - 1 \\ &= 2^{2k+2} - 2\sum_{i=0}^{k-1} 2^{2k-2i} - 2\left(2^{2k-2k}\right) - 1 \\ &= 2^{2k+2} - \sum_{i=0}^{k-1} 2^{2k-2i+1} - 2 - 1 \\ &= 2^{m+1} - \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor - 1} 2^{(m+1)-1-2i} - 2^{(m+1)-1-2\lfloor \frac{m+1}{2} \rfloor}. \end{aligned} \tag{3.3}$$

If m=2k+2, then

$$\begin{aligned} A_{m+1} &= A_{2k+3} \\ &= 2A_{2k+2} \\ &= 2\left(2^{2k+2} - \sum_{i=0}^{(k+1)-1} 2^{2k+2-1-2i} - 2^{2k+1-2(k+1)} \right) \\ &= 2^{2k+3} - \sum_{i=0}^{k} 2^{2k+2-2i} - 2\left(2^{-1}\right) \\ &= 2^{m+1} - \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor - 1} 2^{(m+1)-1-2i} - 2^{(m+1)-1-2\lfloor \frac{m+1}{2} \rfloor}. \end{aligned} \tag{3.4}$$

Therefore, (3.2) holds in both cases.

Continued application of (3.3) and (3.4) indicates that A_n satisfies the third order linear homogeneous recurrence relation

$$A_n = 2A_{n-1} + A_{n-2} - 2A_{n-3}, n \geq 3, \tag{3.5}$$

with characteristic equation

$$x^3 - 2x^2 - x + 2 = 0. \tag{3.6}$$

4. Some Properties

Since the roots of (3.6) are -1,1,2, it follows that the Binet form of the general term of the Alavi sequence is given by

$$A_n = \tfrac{1}{3}2^n + \tfrac{1}{6}(-1)^n + \tfrac{1}{2}. \tag{4.1}$$

An analog of Simson's identity

$$\begin{vmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{vmatrix} = (-1)^n, \tag{4.2}$$

is

$$\begin{vmatrix} A_{n-2} & A_{n-1} & A_n \\ A_{n-1} & A_n & A_{n+1} \\ A_n & A_{n+1} & A_{n+2} \end{vmatrix} = (-2)^n. \tag{4.3}$$

This follows from repeated matrix multiplication:

$$\begin{bmatrix} A_{n-2} & A_{n-1} & A_n \\ A_{n-1} & A_n & A_{n+1} \\ A_n & A_{n+1} & A_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} A_{n-3} & A_{n-2} & A_{n-1} \\ A_{n-2} & A_{n-1} & A_n \\ A_{n-1} & A_n & A_{n+1} \end{bmatrix}$$

since

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -2.$$

5. Concluding Comments

Other properties can be readily developed, For instance, from Equation (4.1) of [6]:

$$A_{n+2} = 1 + \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} (-1)^r \binom{n-m-2r}{m+r} \binom{m+r}{r} 2^{n-2(m+r)}, \tag{5.1}$$

in which the binomial coefficients in the summations, call them $B_{n,m,r}$ satisfy the third order partial recurrence relation [8]:

$$B_{n,m,r} = B_{n-1,m,r} + B_{n-2,m-1,r} + B_{n-3,m,r-1} \tag{5.2}$$

with boundary conditions

$$B_{0,\,m,\,r} \;=\; 0, B_{n,0,\,r} \;=\; \binom{n-2r}{r}, B_{n,\,m0} \;=\; \binom{n-m}{m}.$$

Equation (5.1) is thus an analog of

$$F_{n+1} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m}. \tag{5.3}$$

A topic for further research would be to analyse the following modification:

$$a_1, a_2, ..., a_n, a_1 + a_2, a_1 + a_3, ..., a_{n-1} + a_n, 2a_1 + a_2 + a_3, ...;$$

that is, the sequence formed as follows:

$$\begin{array}{cc|cc}
1\,0\,0\ldots0\,0 & 1\,0\,1\ldots0\,0 & a_1 & a_1+a_3 \\
0\,1\,0\ldots0\,0 & \cdots & a_2 & \cdots \\
\cdots & 0\,0\,0\ldots1\,1 & \cdots & a_1+a_n \\
0\,0\,0\ldots0\,1 & 2\,1\,1\ldots0\,0 & a_n & 2a_1+a_2+a_3 \\
1\,1\,0\ldots0\,0 & \cdots & a_1+a_2 & \cdots
\end{array}$$

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