1 Introduction

In a series of papers the author studied some properties of the well-known arithmetic functions $\varphi$ and $\sigma$ (see, e.g. [5, 7]). With respect to these research, a new function was defined in [1, 2] and some of its properties were discussed there. Here we shall continue this research.

Firstly, following [1, 2], for the natural number $n \geq 2$:

$$ n = \prod_{i=1}^{k} p_i^{\alpha_i}, $$

where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers, we shall define the following functions:

$$ \zeta(n) = \sum_{i=1}^{k} \alpha_i p_i, \quad \zeta(1) = 1, $$

$$ \text{set}(n) = \{p_1, p_2, \ldots, p_k\}, \quad \text{set}(1) = \{1\}, $$

$$ \text{cas}(n) = k, \quad \text{cas}(1) = 1, $$

$$ \text{dim}(n) = \sum_{i=1}^{k} \alpha_i, \quad \text{dim}(1) = 1. $$

2 New properties of function $\zeta$

**Theorem 1:** For every natural number $n$

$$ \zeta(n)\text{cas}(n) \leq n + 4 \quad (1) $$

**Proof** Let $n$ be a prime number. Then, from (1) we obtain:

$$ n + 4 - \zeta(n)\text{cas}(n) = n + 4 - n.1 = 4 \geq 0. $$

Let us assume that (1) is valid for an arbitrary natural number $n$ with $\text{dim}(n) = k$ and let $p$ be an arbitrary prime number. For $p$ there are two cases.
Case 1: \( p \in \text{set}(n) \). Therefore, \( \text{cas}(np) = \text{cas}(n) \) and

\[
np + 4 - \zeta(np)\text{cas}(np) = n.p + 4 - (\zeta(n) + p)\text{.cas}(n)
\]

\[
= n.(p - 1) + n + 4 - \zeta(n)\text{.cas}(n) - p\text{.cas}(n)
\]

(from (1))

\[
\geq n.(p - 1) - p\text{.cas}(n)
\]

\[
= \frac{n}{2}.2.(p - 1) - p\text{.cas}(n) \geq 0,
\]

because, obviously, for every natural number \( n: n \geq 2\text{cas}(n) \) and for every prime number \( p: 2(p - 1) \geq p \).

Case 2: \( p \not\in \text{set}(n) \). Therefore, \( \text{cas}(np) = \text{cas}(n) + 1 \). If \( n \) is a prime number, then

\[
np + 4 - \zeta(np)\text{cas}(np) = n.p + 4 - 2.(n + p) \geq 0
\]

and we obtain “=” only for \( n = 2 \).

Let \( \dim(n) \geq 2 \). Then,

\[
np + 4 - \zeta(np)\text{cas}(np) = n.p + 4 - (\zeta(n) + p).(\text{cas}(n) + 1).
\]

If \( p = 2 \), then by the condition that \( p \not\in \text{set}(n) \), it follows that \( n \) is an odd number. Therefore,

\[
np + 4 - \zeta(np)\text{cas}(np) = 2n + 4 - (\zeta(n) + 2).(\text{cas}(n) + 1)
\]

\[
= 2n + 4 - \zeta(n)\text{cas}(n) - \zeta(n) - 2\text{.cas}(n) - 2
\]

(from (1))

\[
\geq n - \zeta(n) - 2\text{.cas}(n) - 2
\]

(for each natural number \( n \geq 31 \) we can check that: \( \zeta(n) \geq 2\text{.cas}(n) + 6)\)

\[
\geq n - 2\zeta(n) + 4 \geq 0
\]

(by assumption, \( n = q.m \), where \( q \geq 3 \) is a prime number and by inductive assumption \( m + 4 - \zeta(m)\text{cas}(m) \geq 0)\)

\[
= q.m - 2\zeta(q.m) + 4 = q.m - 2.(q + m) + 4
\]

\[
= q.(m - 2) - 2.m + 4 \geq 3.(m - 2) - 2.m + 4 = m - 2 \geq 0.
\]

Finally, let \( p > 2 \). Then,

\[
np + 4 - \zeta(np)\text{cas}(np) = n(p - 1) + n + 4 - (\zeta(n) + p).(\text{cas}(n) + 1)
\]

\[
= n(p - 1) + n + 4 - \zeta(n)\text{cas}(n) - \zeta(n) - p\text{.cas}(n) - p
\]

(from (1))

\[
\geq n(p - 1) - \zeta(n) - p\text{.cas}(n) - p
\]
(we can directly check that $\zeta(n) \geq \text{cas}(n) + 1$

$$\geq n(p - 1) - (p + 1).\zeta(n) \geq 0,$$

because, obviously, for every natural number $n$: $n \geq 2.\zeta(n)$ and for every prime number $p \geq 3$: $2(p - 1) \geq p + 1$.

Therefore, for $n \geq 31$ the assertion is valid. By a direct check we see that

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Hence, Theorem 1 is valid for every natural number $n$.

If we like to construct an inequality with the same components, but in the opposite direction, it could have the form

$$\zeta(n)\text{cas}(n) \geq n,$$

but there are counterexamples, e.g.

$$\zeta(1000)\text{cas}(1000) = (3.2 + 3.5)^2 = 21^2 < 1000.$$

One of the possible inequalities is discussed in the following

**Theorem 2:** For every natural number $n$

$$\zeta(n)^{\text{dim}(n)} \geq n.$$  \hfill (2)

**Proof** Let $n$ be a prime number. Then, from (2) we obtain:

$$\zeta(n)^{\text{dim}(n)} - n = n^1 - n = 0.$$

Let us assume that (2) is valid for an arbitrary natural number $n$ with $\text{dim}(n) = k$, and let $p$ be an arbitrary prime number. Then,

$$\zeta(n.p)^{\text{dim}(n.p)} - n.p = (\zeta(n) + p)^{\text{dim}(n)+1} - n.p$$

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≥ (ζ(n) + p)^{\text{dim}(n)+1} - ζ(n)^{\text{dim}(n)}p
= ζ(n)^{\text{dim}(n)}((ζ(n) + p)(1 + \frac{p}{ζ(n)})^{\text{dim}(n)} - p)
> ζ(n)^{\text{dim}(n)}(ζ(n) + p - p) > 0.

Therefore, (2) is valid.

3 Function ζ and n-th prime number

Following idea from [3, 4], where we introduced four new formulae for the well-known function π(n) (see, e.g. [5, 7]) and a new formula for the n-th prime number p_n, now we shall introduce another (simpler) formula for π(n) and p_n.

Let us define functions sg and s\overline{g} by:

\[
sg(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
1, & \text{if } x > 0
\end{cases},
\]

\[
\overline{sg}(x) = \begin{cases} 
0, & \text{if } x \neq 0 \\
1, & \text{if } x = 0
\end{cases},
\]

where x is a real number.

THEOREM 3: The following equality holds for every natural number n ≥ 2:

\[
π(n) = \sum_{k=2}^{n} \overline{sg}(k - ζ(k)).
\]

Proof: For every natural number k, such that k ≤ n, if k is prime, then k = ζ(k) and hence \(\overline{sg}(k - ζ(k)) = 1\). On the other hand, if k is not prime, then \(k - η(k) > 0\), i.e., \(\overline{sg}(k - η(k)) = 0\). Therefore, the sum is equal to π(n).

Of course, π(0) = 0 and π(1) = 0.

For the so constructed formula for π(n), we can prove by analogy with [3, 4] the following

THEOREM 4: For every natural number n:

\[
p_n = \sum_{i=0}^{C(n)} sg(n - π(i)),
\]

where (see [6])

\[
C(n) = \left\lfloor \frac{n^2 + 3n + 4}{4} \right\rfloor.
\]
References


