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On the solvability of homogeneous two-sided systems in max-algebra

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Abstract

Let $a \oplus b = max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ and extend the pair of operations to matrices and vectors in the same way as in linear algebra. The homogeneous two-sided system in max-algebra is of the form $A \otimes x = B \otimes x$. No polynomial method for solving homogeneous system is known. In this paper, we consider homogeneous two-sided linear systems in max-algebra in a special case. We show that it can be checked in $O(n^3)$ time whether a given two-sided homogeneous system belongs to this special case. Solvability can be decided in $O(n^3)$ time and in the positive case a solution can be found in $O(n^3)$.

1 Introduction

Let $a \oplus b = max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup -\infty$. The component $-\infty$ is a neutral element for $\oplus$ and a null for $\otimes$. We will denote throughout this paper $-\infty$ by ε and for convenience we also denote by the same symbol any vector or matrix whose every component is $-\infty$.

Max-algebra is an analogue of linear algebra developed for the pair of operations $\oplus$ (plus) and $\otimes$ (times), extended to matrices and vectors. That is if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. Also, if $\alpha \in \mathbb{R}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. Max-algebra has been studied by many authors and the reader is referred to [11], [12], [18], [2], [5] or [1].

A system of the form

$$A \otimes x = B \otimes x,$$

where $A = (a_{ij}), B = (b_{ij}) \in \overline{\mathbb{R}}^{m \times n}$, is called *homogeneous two-sided max-linear system* or just *two-sided max-linear system*. General two-sided linear systems in max-algebra have been investigated in several articles e.g [9, 13, 15, 25] and also [4]. A general solution method was presented in [25]. This method uses for its basic solution component, the max-plus closure operation and solves a series of subsystems with decreasing maximum solution. Though no complexity bound was presented for this method. An elimination method for solving $A \otimes x = B \otimes x$ was presented in [9]. It was also shown that the solution set is generated by a finite number of vectors. In [15] a general iterative approach has been developed which assumes that finite upper and lower bounds for all variables be given. The iterative method makes it possible to find an approximation of the maximum solution to the given system, which satisfies the given lower and upper bounds or to find out that no such solution exists. A pseudopolynomial algorithm for solving $A \otimes x = B \otimes y$ was presented in [15]. The algorithm converges to a finite solution from any finite starting point whenever a finite solution exists. This method has been generalized to the systems $A^1 \otimes x^1 = \cdots = A^k \otimes x^k$ [23]. Also the known convergence results are extended in this general settings to the case when the entries of matrices are real. Our aim is to consider homogeneous max-linear system in one special case. We show that if a system belongs to this special case solvability can be decided in $O(n^3)$ time and in the positive case a solution can be found in $O(n^3)$.

2 Definitions and problem formulation

Given $A, B \in \overline{\mathbb{R}}^{m \times n}$, the homogeneous max-linear system is of the form

$$A \otimes x = B \otimes x. \tag{1}$$

Homogeneous max-linear systems always have a solution which is ε, this solution will be called *trivial* and all others *nontrivial*. In what follows we will consider nontrivial solutions only. When the two-sided max-linear system (1) is solvable we write (A, B) is solvable. We will discuss (1) in a special case and present necessary and sufficient conditions for the solvability of such systems. We also present a polynomial algorithm for checking this condition. Recall that if $A \in \mathbb{R}^{n \times n}$ then

$$(A \otimes x)_i = \sum_{m=1}^{n} (a_{im} + x_m).$$

An ordered pair $D = (N, F)$ is called a *digraph* if N is a non-empty set (of *nodes*) and $F \subseteq N \times N$ (the set of *arcs*). A sequence $\pi = (v_1, \ldots, v_p)$ of nodes is called a *path* (in D) if $p = 1$, or $p > 1$ and $(v_i, v_{i+1}) \in F$ for all $i = 1, \ldots, p-1$. The node v_1 is called the *starting node* and v_p the *end node* of π, respectively. The number $p - 1$ is called the *length* of p and will be denoted by $l(\pi)$. If there is a path in D with starting node t and end node u then we say that u is *reachable* from t denoted by $t \to u$. Thus $t \to t$ for any $t \in N$. A digraph D is called *strongly connected* if $t \to u$ for all nodes t, u in D. A path $(v_1, \ldots, v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover $v_i \neq v_j$ for $i, j = 1, \ldots, p-1$, $i \neq j$. The arcs $(v_i, v_{i+1}) \in F$ for $i = 1, \ldots, p-1$ are called the *arcs of the cycle*. We assume that $n \geq 1$ is a given integer and denote by $N = \{1, \ldots, n\}$. The digraph associated with $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is

$$D_A = (N, \{(i, j); a_{ij} > \varepsilon\}).$$

The matrix A is called *irreducible* if D_A is strongly connected and *reducible* otherwise.

3 Max-algebraic eigenvector eigenvalue problem

The *max-algebraic eigenvector-eigenvalue problem* or just *eigenproblem* (EP) is the following: Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $\lambda \in \overline{\mathbb{R}}$ *(eigenvalues)* and $x \in \overline{\mathbb{R}}^{n}$, $x \neq \varepsilon$ *(eigenvectors)* such that

$$A \otimes x = \lambda \otimes x. \tag{2}$$

The theory of max-algebraic eigenproblem is well known [11], [24],[21]. In this section we will give an overview of some results which will be useful in the forthcoming sections.

The set of all eigenvalues and eigenvectors of A will be denoted by $\Lambda(A)$ and $V(A)$, respectively. If $\pi = (i_1, \ldots, i_p)$ is a path in D_A then the *weight* of π is $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{p-1} i_p}$ if $p > 1$ and ϵ if $p = 1$. We denote by $\lambda(A)$ the *maximum cycle mean* of A, that is if D_A has at least one cycle then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{3}$$

where the maximisation is taken over all cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{l(\sigma)} \tag{4}$$

denotes the *mean* of the cycle $\sigma = (i_1, \ldots, i_k, i_1)$. If D_A is acyclic we set $\lambda(A) = \varepsilon$. We denote

$$\begin{aligned} \Lambda(A) = \{&\lambda \in \overline{\mathbb{R}}; \left(\exists x \in \overline{\mathbb{R}}^{n} - \{\varepsilon\}\right) \\ &A \otimes x = \lambda \otimes x\} \\ V(A) = \{&x \in \overline{\mathbb{R}}^{n} - \{\ \varepsilon\}; \exists \lambda \in \overline{\mathbb{R}}, \\ &A \otimes x = \lambda \otimes x\}. \end{aligned}$$

3.1 The eigenvalue

The eigenproblem may have up to n eigenvalues [8]. It is also known that if a matrix A is irreducible then the maximum cycle mean of A is the unique eigenvalue of A. By this, therefore we know that in our case the eigenvalue is unique since our matrix is of the form $A \in \mathbb{R}^{n\times n}$ and hence irreducible.

Theorem 3.1. [11]
Let $A \in \mathbb{R}^{n\times n}$. Then (2) admits a unique eigenvalue $\lambda(A)$ given by

$$\lambda(A) = \max \frac{a_{i_1 i_2} + \cdots + a_{i_{p-1} i_p} + a_{i_p i_1}}{p} \tag{5}$$

where the maximum is taken over all tuples $(i_1, \ldots, i_p)$ for which the indices $i_1, i_2, \ldots, i_p$ are distinct, and $p = 1, 2, \ldots, n$.

Various algorithms for finding $\lambda(A)$ for a given $A \in \overline{\mathbb{R}}^{n\times n}$ exist [19, 10, 16, 17] and [11]. The most efficient among them is Karp's algorithm [19] of complexity bound $O(n^3)$. We will now consider some properties of matrices when their maximum cycle mean is 0. A matrix $A \in \mathbb{R}^{n\times n}$ is called *definite* if $\lambda(A) = 0$.

Proposition 3.1. [11]
Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $\lambda(A)$ is the maximum cycle mean of A then the matrix $\lambda(A)^{-1} \otimes A$ is definite.

Let $\alpha \in \mathbb{R}$ then we have the following

$$\begin{aligned} A \otimes x &= \lambda \otimes x \\ \Longrightarrow (\alpha \otimes A) \otimes x &= (\alpha \otimes \lambda) \otimes x \\ \Longrightarrow V(\alpha \otimes A) &= V(A). \end{aligned}$$

By this and also Proposition 3.1 it is therefore sufficient to show how to solve the eigenproblem for definite matrix.

3.2 The eigenvectors

Let $A \in \mathbb{R}^{n\times n}$. Consider the following matrices obtained by taking powers of A:

$$\begin{aligned} A^{(2)} &= A \otimes A = \\ (a_{ij}^{(2)}) &= \left(\sum_{k}^{\oplus} a_{ik} \otimes a_{kj}\right) \\ &= \left(\max_{k=1,\dots,n} (a_{ik} + a_{kj})\right), \end{aligned}$$

and

$$\begin{aligned} A^{(p)} &= A \otimes A^{(p-1)} = \\ (a_{ij}^{(p)}) &= \sum_{k_{p-1}}^{\oplus} a_{ik} \otimes a_{kj}^{(p-1)} \\ &= \sum_{1\le k_1,\dots,k_{p-1}\le n}^{\oplus} (a_{ik_{p-1}} \otimes \dots \otimes a_{k_1 j}) \\ &= \max_{1\le k_1,\dots,k_{p-1}} (a_{ik_{p-1}} + \dots + a_{k_1 j}). \end{aligned}$$

Recall that by D_A we mean the digraph associated with A. Therefore, in D_A we obtain that a_{ij} is the weight of the path of length 1 from v_i to v_j. It can be observed that $a_{ij}^{(2)}$ is the weight of the longest path of length 2 from v_i to v_j. Also $a_{ij}^{(p)}$ is the weight of the longest path of length p from v_i. The *metric matrix* or *weak transitive closure* of A is defined as

$$\Delta(A) = A \oplus A^{(2)} \oplus A^{(3)} \oplus \dots$$

.

Theorem 3.2. [11]
Suppose $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is definite then

$$\Delta(A) = A \oplus A^{(2)} \oplus \dots \oplus A^{(n)}.$$

Note that if A is definite then $\Delta(A)$ can be found in $O(n^3)$ time using Floyd-Warshall algorithm [22].

Theorem 3.3. [11]
If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, then $\lambda = \lambda(A)$ is the unique eigenvalue of A and every column of $\Delta(\lambda^{-1} \otimes A)$ with a zero diagonal element is an eigenvector of A.

Theorem 3.4. [11]
If N_0 is the set of columns of $\Delta(\lambda^{-1} \otimes A)$ with a zero diagonal element then the set of all eigenvectors for (2) is

$$V(A) = \left\{ \sum\nolimits^{\oplus} \alpha \otimes g_i ; g_i \in N_0, \alpha \in \mathbb{R} \right\}.$$

4 Generalized max-algebraic eigenvector eigenvalue problem

The *generalized max-algebraic eigenvector eigenvalue problem* or just *generalized eigenproblem (GEP)* is: Given $A, B \in \overline{\mathbb{R}}^{n \times n}$, find all $\lambda \in \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$ such that

$$A \otimes x = \lambda \otimes B \otimes x. \tag{6}$$

For the GEP we denote by λ and x the generalized eigenvalue and eigenvectors respectively. Also we denote

$$\begin{aligned} \Lambda(A, B) = \{\lambda \in \overline{\mathbb{R}}; \big(\exists x \in \overline{\mathbb{R}}^n - \{\varepsilon\}\big) \\ A \otimes x = \lambda \otimes B \otimes x\} \\ V(A, B) = \{x \in \overline{\mathbb{R}}^n - \{\ \varepsilon\}; \exists \lambda \in \overline{\mathbb{R}}, \\ A \otimes x = \lambda \otimes B \otimes x\}. \end{aligned}$$

The generalized eigenproblem has been analysed in [3] and [14]. A method for narrowing the search for the generalized eigenvalues are presented in [6] In [14] a number of solvability conditions for general matrices and also solution methods for GEP in some special cases have been presented. We will now present the existence and uniqueness of eigenvalues for GEP discussed in [3] in one special case. In what follows we assume that the matrices A and B are finite. Recall that $N = \{1, \dots, n\}$ and define

$$(c_1(j, i), c_2(j, i), \dots) = c(j, i) = A_j - B_i,$$

the vector obtained by subtracting the i^{th} column of B from the j^{th} column of A for all $i, j \in N$. Also we denote by $c_k(j, i)$ for $k \in N$ the k^{th} component of $c(j, i)$.

Lemma 4.1. [3]
Let $x \in V(A, B)$, $i \in N$ and suppose that $j, k \in N$ satisfy $(A \otimes x)_i = a_{ij} + x_j$, $(B \otimes x)_k = b_{ki} + x_i$. Then $c_k(j, i) \leq c_i(j, i)$.

Using Lemma 4.1 the following condition is defined as *condition (T)* for all $i, j, k \in N$ with $k \neq i, k \neq j$,

$$c_k(j,i) \geq c_i(j,i). \tag{7}$$

This condition is called *strict (T)* if the inequality (7) is strict [3].

Before discussing the uniqueness of generalized eigenvalues we define the following condition [3]: *Strong (T)* is a condition when strict (T) holds and (7) holds with strict inequality also for $j = k \neq i$.

Theorem 4.1. [3]
If A, B satisfy strict (T) then $V(A,B) \neq \emptyset$ and also $\Lambda(A,B) \neq \emptyset$.

Theorem 4.2. [3]
Let A, B satisfy strong(T). Then $|\Lambda(A,B)| = 1$ and the unique generalized eigenvalue is

$$\max \frac{(a_{i_1 i_2} + \cdots + a_{i_p i_1}) - (b_{i_1 i_1} + \cdots + a_{i_p i_p})}{p} \tag{8}$$

where the maximum is taken over all tuples $(i_1, i_2, \ldots, i_p)$ for which the indices $i_1, i_2, \ldots, i_p$ are distinct, and $p = 1, 2, \ldots, n$.

Given $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$ we denote by $\lambda(A,B)$ the unique eigenvalue of (6). Also, we define $C = (c_{ij}) \in \mathbb{R}^{n \times n}$ as a matrix whose entry in the ith row and jth column is $a_{ij} \otimes b_{ii}^{-1}$. That is $C = (c_{ij}) = (a_{ij} \otimes b_{ii}^{-1})$. The eigenproblem associated with C is the following

$$C \otimes x = \lambda \otimes x. \tag{9}$$

It follows from Theorem 3.3 that $\lambda(C)$ is the unique eigenvalue of C. The following theorem shows that a *eigenvector* and *eigenvalue* (that is $\lambda \neq \varepsilon, x \neq \varepsilon$) for GEP exists whenever a strict (T) condition is satisfied.

Theorem 4.3.
If A, B satisfy strong(T) then $V(A,B) = \{x \in \overline{\mathbb{R}}^n, x \neq \varepsilon; C \otimes x = \lambda(A,B) \otimes x\}$ where $C = (c_{ij}) = (a_{ij} \otimes b_{ii}^{-1})$.

Proof.
Suppose that A and B satisfy strong (T). It follows from Theorem 4.1 that $V(A,B) \neq \emptyset$ and also $\Lambda(A,B) \neq \emptyset$. Let $\lambda = \lambda(A,B)$ with $x \in V(A,B)$. Now let $k \in N$ and select i such that $(B \otimes x)_k = b_{ki} + x_i$. Similarly, select j such that $(A \otimes x)_i = a_{ij} + x_j$. It follows from Lemma 4.1 that $c_k(j,i) \leq c_i(j,i)$. This contradicts the assumption that A, B satisfy strong (T) if $i \neq k$. Therefore we have $i = k$ and

$$(B \otimes x)_k = b_{kk} + x_k, \; k \in N.$$

Consequently,

$$C \otimes x = \lambda \otimes x, \tag{10}$$

where $C = (c_{ij}) \in \mathbb{R}^{n \times n}$ and $c_{ij} = a_{ij} \otimes b_{ii}^{-1}$. Since $|\Lambda(A,B)| = 1$ (Theorem 4.2) we have $\lambda = \lambda(A,B)$ and thus $V(A,B) \subseteq \{x \in \overline{\mathbb{R}}^n, x \neq \varepsilon; C \otimes x = \lambda(A,B) \otimes x\}$. $V(A,B) \supseteq \{x \in \overline{\mathbb{R}}^n, x \neq \varepsilon; C \otimes x = \lambda(A,B) \otimes x\}$ by similar argument. $\square$

Corollary 4.1.
Let A, B satisfy strong(T). Then $\lambda(A, B) = \lambda(C)$.

Proof.
The proof follows from Theorems 3.1 and 4.3. □

5 Necessary solvability conditions for $A \otimes x = B \otimes x$

In this section use the results presented in the previous section and present a solvability condition for two-sided max-linear systems (1) where A and B are finite square matrices. We also present an $O(n^3)$ algorithm for checking the solvability of such systems.

Due to Theorem 4.2 we know that GEP has a unique eigenvalue if the strong (T) condition is satisfied. For this case also, Theorem 4.3 describes all the eigenvectors of GEP. Therefore we have the following:

Corollary 5.1.
Let A, B satisfy strong(T). Then (A, B) is solvable if and only if C is definite.

Proof.
Suppose A, B satisfy strong(T). (A, B) solvable implies that GEP is solvable with $\lambda = 0$. But unique generalized eigenvalue is denoted as $\lambda(A, B)$ and given by (8). Therefore, $\lambda(A, B) = \lambda(C) = 0$ (Corollary 4.1).
Similarly, A and B satisfy strong (T), $\lambda(C) = 0$ imply that $|\Lambda(A, B)| = 1$ and $\lambda(A, B) = \lambda(C) = 0$ (Corollary 4.1). Consequently, (A, B) is solvable. □

Solvability of (1) is equivalent to checking whether $0 \in \Lambda(A, B)$. For A, B satisfying strong (T) the unique generalized eigenvalue is given by (8) we will present a polynomial algorithm for checking whether given matrices A and B satisfy strong (T) condition. If strong (T) is satisfied we obtain matrix C from A and B and in this case $\lambda(C)$ is the unique generalized eigenvalue (Corollary 4.1). Hence if A, B satisfy strong (T) it remains to check whether $\lambda(C) = 0$. This can be done using Karp's algorithm in $O(n^3)$ time applied to C. If $\lambda(C) = 0$ then Theorem 4.3 describes all the eigenvectors satisfying (6) and thus all solutions to the system $A \otimes x = B \otimes x$ can be found by using the Floyd-Warshall algorithm applied to C with computational complexity $O(n^3)$.

Theorem 5.1. Algorithm STRONG-(T) is correct and its computational complexity is $O(n^3)$.

Proof. The correctness follows from the definition of strong (T). Since $i, j \in N$ then the main loop is repeated $O(n^3)$ times. The value c can be found in $O(n)$ time. The inner loop is repeated n times. Thus the computational complexity of Algorithm STRONG-(T) is $O(n^3)$. □

Algorithm 1 STRONG-T (Strong (T) condition for the homogeneous two-sided systems)

Input: $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$

Output: $flag = true$ if the condition is satisfied, $flag = false$ otherwise

1. Set $flag := true$
2. for all $i, j = 1, \ldots, n$ do
   begin
   $c := c(j, i) = A_j - B_i$
   for $k = 1, \ldots, n$ do
   if $k \neq i$ then
   begin
   if $c_k \leq c_i$ then $flag := false$
   stop
   end
   end

6 An example

Consider the homogeneous two-sided system in which

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & -3 & -3 \\ 2 & 4 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 4 & 1 \\ -3 & 4 & -3 \\ -3 & 1 & 5 \end{pmatrix}. \quad (11)$$

By running Algorithm STRONG-T the system satisfies the strong (T) condition. The run of the algorithm is summarised in the following table. The corresponding matrix C matrix is

i, j	$c(j, i) = A_j - B_i$	$k \neq i$, $k \neq j$	$j = k \neq i$
1, 1	$\begin{pmatrix} 0 \\ 6 \\ 5 \end{pmatrix}$	$k = 2$, $c_2 > c_1$ and $k = 3$, $c_3 > c_1$	-
1, 2	$\begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix}$	$k = 3$, $c_3 > c_1$	$k = 2$, $c_2 > c_1$
1, 3	$\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$	$k = 2$, $c_2 > c_1$	$k = 3$, $c_3 > c_1$
2, 1	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$k = 3$, $c_3 > c_2$	$k = 1$, $c_1 > c_2$
2, 2	$\begin{pmatrix} -2 \\ -7 \\ 3 \end{pmatrix}$	$k = 1$, $c_1 > c_2$ and $k = 3$, $c_3 > c_2$	-

i, j	$c(j,i) = A_j - B_i$	$k \neq i$, $k \neq j$	$j = k \neq i$
2, 3	$\begin{pmatrix} -3 \\ -7 \\ -2 \end{pmatrix}$	$k = 1$, $c_1 > c_2$	$k = 3$, $c_3 > c_2$
3, 1	$\begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$	$k = 2$, $c_2 > c_3$	$k = 1$, $c_1 > c_3$
3, 2	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	$k = 1$, $c_1 > c_3$	$k = 2$, $c_2 > c_3$
3, 3	$\begin{pmatrix} 0 \\ 0 \\ -6 \end{pmatrix}$	$k = 1$, $c_1 > c_3$ and $k = 2$, $c_2 > c_3$	-

Table 1: Checking the strong (T) condition for matrices A and B.

$$C = \begin{pmatrix} 0 & -2 & -3 \\ -1 & -7 & -7 \\ -3 & -1 & -6 \end{pmatrix}. \tag{12}$$

It can be identified straightforwardly, that C is definite since all the cycles are less than or equals to zero and one is zero.

Since A and B satisfy the strong(T) condition and the matrix C is definite, we can find all solutions to $A \otimes x = B \otimes x$ by finding the eigenvectors corresponding C. Hence we evaluate the transitive closure $\Delta(C)$. By Theorem 3.3 every column of $\Delta(C)$ with zero diagonal element is an eigenvector of C. The eigenvector found is therefore a solution to the homogeneous system with matrices defined in (11). We evaluate the transitive closure $\Delta(C)$ as follows:

$$\begin{aligned}
\Delta(C) &= C \oplus C^2 \oplus C^3 \\
&= \begin{pmatrix} 0 & -2 & -3 \\ -1 & -7 & -7 \\ -3 & -1 & -6 \end{pmatrix} \oplus \begin{pmatrix} 0 & -2 & -3 \\ -1 & -3 & -4 \\ -2 & -5 & -6 \end{pmatrix} \\
&\oplus \begin{pmatrix} 0 & -2 & -3 \\ -1 & -3 & -4 \\ -2 & -4 & -5 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 & -3 \\ -1 & -3 & -4 \\ -2 & -1 & -5 \end{pmatrix} = (g_{ij}).
\end{aligned}$$

Since $g_{11} = 0$, the first column is an eigenvector of C and hence a solution to $A \otimes x = B \otimes x$. Thus, we conclude that all solutions to the homogeneous two-sided system $A \otimes x = B \otimes x$ are multiples of $x = (0,\ -1,\ -2)^T$.

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