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# A Note on Some Diagonal, Row and Partial Column Sums of a Zeckendorf Triangle

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**Abstract**

This note fleshes out some of the characteristics of what is referred to as a Zeckendorf triangle which is composed of Fibonacci number multiples of the Fibonacci sequence. It arose it arose in an infinite binary matrix related to the Zeckendorf representations of the non-negative integers.

**Keywords:** Fibonacci numbers, convolutions, recurrence relations, Kronecker delta, Zeckendorf representations

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## 1. Introduction

There are many Fibonacci-related triangles in the literature, the most famous of which is the appearance of the Fibonacci numbers along the diagonals of the Pascal triangle [1]. Other appearances [1,2,5,6,8] occur in somewhat unexpected ways. Recently, Griffiths [3] cited the triangle below [10] which arose from an infinite matrix of 0s and 1s where the rows correspond to the Zeckendorf representations of the non-negative integers.

The purpose of this note is to consider the sequences which arise from diagonal and row sums and the partial central column sums of this triangle.

## 2. The Zeckendorf Triangle

Rather than re-label the triangle as a Fibonacci triangle because it has Fibonacci numbers along its left and right edges, it might be appropriate to refer to it as the “Zeckendorf triangle”. It appears below in a left-corrected form because this makes it more obvious that the columns are Fibonacci number multiples of the numbers in the Fibonacci sequence than when the triangle is presented in an isosceles format.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | |
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | | | | | | | | |
| 3 | 2 | 2 | 3 | | | | | | | |
| 5 | 3 | 4 | 3 | 5 | | | | | | |
| 8 | 5 | 6 | 6 | 5 | 8 | | | | | |
| 13 | 8 | 10 | 9 | 10 | 8 | 13 | | | | |
| 21 | 13 | 16 | 15 | 15 | 16 | 13 | 21 | | | |
| 34 | 21 | 26 | 24 | 25 | 24 | 26 | 21 | 34 | | |
| 55 | 34 | 42 | 39 | 40 | 40 | 39 | 42 | 34 | 55 | |
| 89 | 55 | 68 | 63 | 65 | 64 | 65 | 63 | 68 | 55 | 89 |

The column sequences are actually particular cases of the generalized Fibonacci and Lucas sequences $\{F_{m,n}\}$ which satisfy the Fibonacci partial recurrence relation [2]

$$F_{m,n} = F_{m,n-1} + F_{m,n-2},\ m \geq 0, n > 2\,.$$

We now label the sequences of diagonal, row and partial column sums by $\{d_n\}, \{r_n\}, \{c_n\}$, respectively. We observe in turn that

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\{d_n\}$ | 1 | 1 | 3 | 4 | 9 | 13 | 25 | 38 | 68 | 106 | 182 |
| $\{r_n\}$ | 1 | 2 | 5 | 10 | 20 | 38 | 71 | 130 | 235 | 420 | 744 |
| $\{c_n\}$ | 1 | 2 | 6 | 15 | 40 | 104 | 273 | 714 | 1870 | 4895 | 12816 |

In this, the $\{c_n\}$ has been formed from the central column of the original isosceles from of the triangle in [3], namely

$$\{c_n\} \equiv \{z_{1,1}, z_{3,2}, z_{5,3}, z_{7,4}, z_{9,5}, \ldots\} \equiv \{1,1,4,9,25,64,\ldots\} \tag{2.1}$$

in which the $\{z_{i,j}\}$ are the elements of the isosceles form of the Zeckendorf triangle.

The $\{r_n\}$ is a Fibonacci convolution sequence [4] where

$$5r_n = nF_{n+2} + (n+2)F_n\,. \tag{2.2}$$

Then, from the triangle, it can be observed that we get the recurrence relations

$$d_{2n-j} + d_{2n-j+1} + \delta_{1,j}F_{2n-j+1} = d_{2n-j+2},\ j = 0,1, \tag{2.3}$$

in which $\delta_{i,j}$ is the Kronecker delta and $\{F_n\}$ are the Fibonacci numbers; similarly

$$r_n + r_{n+1} + F_{n+2} = r_{n+2} \tag{2.5}$$

and

$$c_n + c_{n+1} + S_{n+1} = c_{n+2} \tag{2.6}$$

where $\{S_n\}$ is the layer susceptibility series for square lattices [1]:

$$\{S_n\} \equiv \{1,3,7,19,49,127,321,813,2041,...\}.$$

We note that (2.5) reduces to (2.2) with repeated use of the Fibonacci recurrence relation:

$$\begin{aligned}
5r_n + 5r_{n+1} + 5F_{n+2} &= nF_{n+2} + (n+2)F_n + (n+1)F_{n+3} + (n+3)F_{n+1} + 5F_{n+2} \\
&= (n+2)F_n + (n+3)F_{n+1} + (n+5)F_{n+2} + (n+1)F_{n+3} \\
&= (n+2)F_n + (n+3)F_{n+1} + 3F_{n+2} - F_{n+3} + (n+2)F_{n+4} \\
&= (n+2)F_n + (n+2)F_{n+1} + 2F_{n+2} + (n+2)F_{n+4} \\
&= (n+4)F_{n+2} + (n+2)F_{n+4} \\
&= 5r_{n+2}.
\end{aligned}$$

## 3, Concluding Comments

For other connections between the Fibonacci numbers and the Zeckendorf representations of the integers see [9]. The row numbers, $\{r_n\}$, are shown by Griffiths to be convolutions of the Fibonacci numbers. Early identities for the convolutions of the Fibonacci numbers were developed by Riordan [7].

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