Abstract. Two new sequences from Fibonacci type are introduced and the explicit formulae for their $n$-th members are given.

Keywords: Fibonacci sequence, 2-Fibonacci sequence

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In [2, 4, 5, 6] four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described and called 2-Fibonacci sequences (or 2-F-sequences). On their base, in [3] the following two new schemes are introduced.

$$\begin{align*}
\alpha_0 &= 2a, \; \beta_0 = 2b, \; \alpha_1 = 2c, \; \beta_1 = 2d \\
\alpha_{n+2} &= \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n, \; n \geq 0 \\
\beta_{n+2} &= \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n, \; n \geq 0
\end{align*}$$

and

$$\begin{align*}
\alpha_0 &= 2a, \; \beta_0 = 2b, \; \alpha_1 = 2c, \; \beta_1 = 2d \\
\alpha_{n+2} &= \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n, \; n \geq 0 \\
\beta_{n+2} &= \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n, \; n \geq 0
\end{align*}$$

Let $\sigma$ be the integer function defined for every $k \geq 0$ by:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma(4.k + r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
</tr>
</tbody>
</table>
Obviously, for every \( n \geq 0 \),
\[
\sigma(n + 2) + \sigma(n) = 0.
\]

In [3] the following two assertions are formulated and proved for these two sequences.

**THEOREM 1.** For every natural number \( n \geq 0 \)
\[
\alpha_{n+2} = (F_{n+1} + \sigma(n - 1)).a + (F_{n+1} + \sigma(n + 1)).b + (F_{n+2} + \sigma(n + 2)).c + (F_{n+2} + \sigma(n)).d
\]
\[
\beta_{n+2} = (F_{n+1} + \sigma(n + 1)).a + (F_{n+1} + \sigma(n - 1)).b + (F_{n+2} + \sigma(n)).c + (F_{n+2} + \sigma(n + 2)).d.
\]

**THEOREM 2.** For each natural number \( n \geq 0 \)
\[
\alpha_{n+2} = (F_{n+1} + \rho(n)).a + (F_{n+1} - \rho(n)).b + (F_{n+2} + \rho(n + 1)).c + (F_{n+2} - \rho(n + 1)).d
\]
\[
\beta_{n+2} = (F_{n+1} - \rho(n)).a + (F_{n+1} + \rho(n)).b + (F_{n+2} - \rho(n + 1)).c + (F_{n+2} + \rho(n + 1)).d.
\]

Now, we will introduce two new schemes. The first one is:

\[
\alpha_0 = 2a, \quad \beta_0 = 2b, \quad \alpha_1 = 2c, \quad \beta_1 = 2d
\]
\[
\alpha_{n+2} = \beta_{n+1} + \frac{\alpha_n + \beta_n}{2}, \quad n \geq 0
\]
\[
\beta_{n+2} = \alpha_{n+1} + \frac{\alpha_n + \beta_n}{2}, \quad n \geq 0
\]

where \( a, b, c, d \) are given constants.

If we set \( a = b \) and \( c = d \), then sequences \( \{\alpha_i\}_{i=0}^{\infty} \) and \( \{\beta_i\}_{i=0}^{\infty} \) will coincide with each other and with the sequence \( \{F_i\}_{i=0}^{\infty} \), which is called a generalized Fibonacci sequence, where
\[
F_0(a, c) = a,
\]
\[
F_1(a, c) = c,
\]
\[
F_{n+2}(a, c) = F_{n+1}(a, c) + F_n(a, c).
\]

Let \( F_i = F_i(0, 1); \{F_i\}_{i=0}^{\infty} \) be the ordinary Fibonacci sequence.

The first 10 members of the first of the new schemes have the form shown on Table 1.

<table>
<thead>
<tr>
<th></th>
<th>( \alpha_n )</th>
<th>( \beta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2a</td>
<td>2b</td>
</tr>
<tr>
<td>1</td>
<td>2c</td>
<td>2d</td>
</tr>
<tr>
<td>2</td>
<td>( a + b + 2d )</td>
<td>( a + b + 2c )</td>
</tr>
<tr>
<td>3</td>
<td>( a + b + 3c + d )</td>
<td>( a + b + c + 3d )</td>
</tr>
<tr>
<td>4</td>
<td>( 2a + 2b + 2c + 4d )</td>
<td>( 2a + 2b + 4c + 2d )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3a + 3b + 6c + 4d</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>------------------</td>
</tr>
<tr>
<td>5</td>
<td>5a + 5b + 7c + 9d</td>
<td>5a + 5b + 9c + 7d</td>
</tr>
<tr>
<td>6</td>
<td>8a + 8b + 14c + 12d</td>
<td>8a + 8b + 12c + 14d</td>
</tr>
<tr>
<td>7</td>
<td>13a + 13b + 29c + 22d</td>
<td>13a + 13b + 22c + 20d</td>
</tr>
<tr>
<td>8</td>
<td>21a + 21b + 35c + 33d</td>
<td>21a + 21b + 33c + 35d</td>
</tr>
</tbody>
</table>

**THEOREM 3.** For every natural number \( n \geq 0 \)

\[
\alpha_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^{n+1}).c + (F_{n+2} + (-1)^n).d
\]

\[
\beta_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^n).c + (F_{n+2} + (-1)^{n+1}).d
\]

The proof of this assertion can be made, for example, by induction.

For \( n = 0 \) we see the validity of the two formulas from Table 1. Let us assume that these formulas are valid for some natural number \( n \geq 0 \). Then, having in mind that for every natural number \( n \geq 0 \)

\[
(-1)^n + (-1)^{n+1} = 0,
\]

we obtain

\[
\alpha_{n+3} = \beta_{n+2} + \frac{\alpha_{n+1} + \beta_{n+1}}{2}
\]

\[
= F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^n).c + (F_{n+2} + (-1)^{n+1}).d
\]

\[
+ \frac{1}{2}(F_n.a + F_n.b + (F_{n+1} + (-1)^n).c + (F_n + (-1)^{n-1}).d
\]

\[
+ F_n.a + F_n.b + (F_{n+1} + (-1)^{n-1}).c + (F_{n+1} + (-1)^n).d)
\]

\[
= F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^n).c + (F_{n+2} + (-1)^{n+1}).d
\]

\[
+ F_n.a + F_n.b + F_{n+1}.c + F_n.d
\]

\[
= F_{n+2}.a + F_{n+2}.b + (F_{n+3} + (-1)^n).c + (F_{n+3} + (-1)^{n+1}).d
\]

\[
= F_{n+2}.a + F_{n+2}.b + (F_{n+3} + (-1)^{n+2}).c + (F_{n+3} + (-1)^{n+1}).d.
\]

The formula for \( \beta_{n+3} \) may be checked in similar manner.

The second new sequence has the form:

\[
\alpha_0 = 2a, \quad \beta_0 = 2b, \quad \alpha_1 = 2c, \quad \beta_1 = 2d
\]

\[
\alpha_{n+2} = \alpha_{n+1} + \frac{\alpha_n + \beta_n}{2}, \quad n \geq 0
\]

\[
\beta_{n+2} = \beta_{n+1} + \frac{\alpha_n + \beta_n}{2}, \quad n \geq 0
\]

where \( a, b, c, d \) are given constants.

The first 10 members of the second of the new schemes have the form shown on Table 2.
THEOREM 4. For each natural number \( n \geq 0 \)

\[
\alpha_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2} + 1).c + (F_{n+2} - 1).d
\]

\[
\beta_{n+2} = F_{n+1}.a + F_{n+1}.b + +(F_{n+2} - 1).c + (F_{n+2} + 1).d.
\]

For \( n = 0 \) we see the validity of the two formulas from Table 2. Let us assume that these formulas are valid for some natural number \( n \geq 0 \). We shall check the validity of the second formula for \( n + 1 \).

\[
\beta_{n+3} = \frac{\alpha_{n+1} + \beta_{n+1}}{2}
\]

\[
= F_{n+1}.a + F_{n+1}.b + (F_{n+2} - 1).c + (F_{n+2} + 1).d
\]

\[
+ \frac{1}{2}(F_{n}.a + F_{n}.b + (F_{n+1} + 1).c + (F_{n+1} - 1).d)
\]

\[
(F_{n}.a + F_{n}.b + (F_{n+1} - 1).c + (F_{n+1} + 1).d)
\]

\[
= F_{n+1}.a + F_{n+1}.b + (F_{n+2} - 1).c + (F_{n+2} + 1).d
\]

\[
+ F_{n}.a + F_{n}.b + F_{n+1}c + F_{n+1}.d
\]

\[
= F_{n+1}.a + F_{n+1}.b + (F_{n+3} - 1).c + (F_{n+3} + 1).d.
\]

The formula for \( \alpha_{n+3} \) may be checked in similar manner.

2. A digital arithmetic function will be described, following [1, 7].

Let

\[
n = \sum_{i=1}^{k} a_i 10^{k-i} \equiv a_1 a_2 \ldots a_k,
\]
where \(a_i\) is a natural number and \(0 \leq a_i \leq 9\) (\(1 \leq i \leq k\)). Let for \(n = 0\) : \(\varphi(n) = 0\) and for \(n > 0\):

\[
\varphi(n) = \sum_{i=1}^{k} a_i.
\]

We shall use the decimal count system everywhere hereafter.

Let us define a sequence of functions \(\varphi_0, \varphi_1, \varphi_2, \ldots\), where (\(l\) is a natural number)

\[
\varphi_0(n) = n,
\]

\[
\varphi_{l+1} = \varphi(\varphi_l(n)).
\]

Obviously, for every \(l \in \mathcal{N}: \varphi_l : \mathcal{N} \to \mathcal{N}\). Since for \(k > 1\)

\[
\varphi(n) = \sum_{i=1}^{k} a_i < \sum_{i=1}^{k} a_i.10^{k-i} = n.
\]

Then for every \(n \in \mathcal{N}, l \in \mathcal{N}\) will exist so that

\[
\varphi_l(n) = \varphi_{l+1}(n) \in \Delta \equiv \{0, 1, 2, \ldots, 9\}.
\]

Let function \(\psi\) be defined by

\[
\psi(n) = \varphi_l(n),
\]

where

\[
\varphi_{l+1}(n) = \varphi_l(n).
\]

Let be given the sequence \(a_1, a_2, \ldots\), with its members being natural numbers and let

\[
c_i = \psi(a_i) \ (i = 1, 2, \ldots).
\]

Hence, we deduce the sequence \(c_1, c_2, \ldots\) from the former sequence. If \(k\) and \(l\) exist so that \(l \geq 0\),

\[
c_{i+l} = c_{k+i+l} = c_{2k+i+l} = \ldots
\]

for \(1 \leq i \leq k\), then we, following [1], shall say that \([c_{l+1}, c_{l+2}, \ldots, c_{l+k}]\) is base of the sequence \(a_1, a_2, \ldots\) with length of \(k\) and with respect to function \(\psi\).

On Tables 3 and 4 we shall show that the two new sequences have bases with length 24.

**Table 3**

<table>
<thead>
<tr>
<th>(\psi(\alpha_n) = \psi(\bullet))</th>
<th>(\psi(\beta_n) = \psi(\bullet))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2a</td>
</tr>
<tr>
<td>1</td>
<td>2c</td>
</tr>
<tr>
<td>2</td>
<td>(a+b+2d)</td>
</tr>
<tr>
<td>3</td>
<td>(a+b+3c+d)</td>
</tr>
<tr>
<td></td>
<td>$\psi(\alpha_n) = \psi(\bullet)$</td>
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<td>-----------------------------------</td>
</tr>
<tr>
<td>0</td>
<td>$2a$</td>
</tr>
<tr>
<td>1</td>
<td>$2c$</td>
</tr>
<tr>
<td>2</td>
<td>$a + b + 2c$</td>
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<tr>
<td>3</td>
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<tr>
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<td>$8a + 8b + 5c + 3d$</td>
</tr>
<tr>
<td>8</td>
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<td>9</td>
<td>$3a + 3b + 8c + 6d$</td>
</tr>
<tr>
<td>10</td>
<td>$7a + 7b + 2d$</td>
</tr>
<tr>
<td>11</td>
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</tr>
<tr>
<td>12</td>
<td>$8a + 8b + 8c + d$</td>
</tr>
<tr>
<td>13</td>
<td>$7d$</td>
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<tr>
<td>14</td>
<td>$8a + 8b + 7c$</td>
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<td>15</td>
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<td>16</td>
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</tr>
<tr>
<td>17</td>
<td>$6a + 6b + 5c + 3d$</td>
</tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>25</td>
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<td>(8a + 8b + c + 8d)</td>
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<tr>
<td>12</td>
<td>(7d)</td>
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<td>(2c)</td>
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**References**


