

Why are some right-end digits absent in primitive Pythagorean triples?

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Abstract: Integer structure analysis in the Ring Z_3 shows that the right-end digit (RED) couples (1,4), (5,6) and (5,0) for x^2, y^2 in the primitive Pythagorean triple (pPt) in the equation $z^2 = x^2 + y^2$ do not lead to the primitive form of triple. The rows of x^2, y^2 with these REDs do not add to the required form for z^2 . Since $3 \nmid z$, the row of z^2 must follow the pentagonal numbers. Common factors for x, y are also inconsistent with pPt formation so that the (x^2, y^2) RED (5,0) may be discarded directly.

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1. Introduction

Odd squares always have a right-end digit (RED) of 1, 5, 9, and even square have REDs of 0, 4, 6. Thus the sum of an odd and even square has the RED pattern shown in Table 1 for

$$(x^2)^* + (y^2)^* = (z^2)^*.$$

$(y^2)^* \downarrow$ $(x^2)^* \rightarrow$	1	5	9
0	1	5	9
4	5	9	3
6	7	1	5

Table 1: Sums of squares of x (odd) and y (even)

Since z will be odd, $(z^2)^* = 1, 5, 9$. Thus, $(9+4)^* = 3$ and $(1+6)^* = 7$ are invalid. Since $(5+6)^*$, $(5+0)^* = 5$ and $(1+4)^* = 5$, these systems could be valid, but however they do not form pPts. RED couples (6,9), (5,4) and (1,0) are represented in pPts, but not (1,4), (5,6) and (5,0) (Table 2). The reasons for this may be understood from integer structure analysis [4]. One of the underlying reasons is that in the modular ring Z_4 (Table 3) odd integers that equate to a

z	x or y	y or x	z^2	x^2 or y^2	y^2 or x^2	x^2 or y^2	REDs
5	4	3	25	16	9	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	6+9=5
13	12	5	169	144	25	$\bar{1}_3 = \bar{0}_3(e) + \bar{1}_3(o)$	6+9=5
17	15	8	289	225	64	$\bar{1}_3 = \bar{0}_3(o) + \bar{1}_3(e)$	5+4=9
25	24	7	625	576	49	$\bar{1}_3 = \bar{0}_3(e) + \bar{1}_3(o)$	6+9=5
29	21	20	841	441	400	$\bar{1}_3 = \bar{0}_3(o) + \bar{1}_3(e)$	1+0=1
37	35	12	1369	1225	144	$\bar{1}_3 = \bar{1}_3(o) + \bar{0}_3(e)$	5+4=9
65	56	33	4225	3136	1089	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	6+9=5
65	63	16	4225	3969	256	$\bar{1}_3 = \bar{0}_3(o) + \bar{1}_3(e)$	6+9=5
73	55	48	5329	3025	2304	$\bar{1}_3 = \bar{1}_3(o) + \bar{0}_3(e)$	5+4=9
85	77	36	7225	5929	1296	$\bar{1}_3 = \bar{1}_3(o) + \bar{0}_3(e)$	9+6=5
137	105	88	18769	11025	7744	$\bar{1}_3 = \bar{0}_3(o) + \bar{1}_3(e)$	5+4=9
169	120	119	28561	14400	14161	$\bar{1}_3 = \bar{0}_3(e) + \bar{1}_3(o)$	0+1=1
205	156	133	42025	24336	17689	$\bar{1}_3 = \bar{0}_3(e) + \bar{1}_3(o)$	9+6=5
221	220	21	48241	48400	441	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	0+1=1
233	208	105	54289	43264	11025	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	4+5=9
305	224	207	93025	50176	42849	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	6+9=5
325	253	204	105625	64009	41616	$\bar{1}_3 = \bar{1}_3(o) + \bar{0}_3(e)$	6+9=5
1233	208	105	54289	43264	11025	$\bar{1}_3 = \bar{1}_3(e) + \bar{0}_3(o)$	4+5=9
1381	1320	931	1907161	1742400	866761	$\bar{1}_3 = \bar{0}_3(e) + \bar{1}_3(o)$	0+1=1
10405	10403	204	14619025	14595409	41616	$\bar{1}_3 = \bar{1}_3(o) + \bar{0}_3(e)$	9+6=5

Table 2: Primitive Pythagorean Triples – o: odd; e: even

Row	Function	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31
8		32	33	34	35
9		36	37	38	39
10		40	41	42	43

Table 3: Modular Ring Z_4 : $\bar{2}_4$ has no powers; $\bar{3}_4$ has no even powers

sum of squares are always confined to Class $\bar{1}_4$ ($4r_1+1$) (first shown by Fermat [1]). If integers in $\bar{1}_4$ have factors from Class $\bar{3}_4$, then the pPt will not form [4,5]. Since z must be a sum of squares, this is a critical structural factor.

Another critical structural factor is that odd squares, N^2 , with $3 \nmid N$ have rows that follow the pentagonal numbers [6]. These factors underlie the failure of the RED couples (1,4), (5,6) and (5,0) to be found in pPts.

2. Modular ring structures

For simplicity we move from Z_4 to Z_3 (Table 4). This modular ring has mixed odd/even classes, but it is the most compact of the rings [7]. The rows of the integers in Z_3 corresponding to the rows in Z_4 are shown in Table 5, and some other relevant properties in Table 6.

Row	Function	$3r_0$	$3r_1+1$	$3r_2+2$
	Class	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$
0		0	1	2
1		3	4	5
2		6	7	8
3		9	10	11
4		12	13	14
5		15	16	17
6		18	19	20
7		21	22	23
8		24	25	26
9		27	28	29
10		30	31	32

Table 4: Modular Ring Z_3

Class in Z_3	Rows of odd integers in Z_3	Rows of even integers in Z_3
$\bar{0}_3, 3 \mid N$	$\frac{1}{3}(4R_1+1) \wedge \frac{1}{3}(4R_3+3)$	$\frac{4}{3}R_0 \vee \frac{1}{3}(4R_2+2)$
$\bar{1}_3, 3 \nmid N$	$\frac{4}{3}R_1 \vee \frac{2}{3}(2R_3+1)$	$\frac{1}{3}(4R_0-1) \vee \frac{1}{3}(4R_2+1)$
$\bar{2}_3, 3 \nmid N$	$\frac{1}{3}(4R_1-1) \vee \frac{1}{3}(4R_3+1)$	$\frac{1}{3}(4R_0-2) \vee \frac{4}{3}R_2$

Table 5: Equivalent rows in Z_4 and $Z_3 - R_j$; row in Z_4

Integer Parity	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$
odd/even	$3 \mid N$	$3 \nmid N$	$3 \nmid N$
odd	odd row	even row	odd row
even	even row	odd row	even row
even powers	yes	yes	no
odd powers	yes	yes	yes

Table 6: Some characteristics of Z_3

3. Formation of pPts

In the equation

$$z^2 = x^2 + y^2, \quad (3.1)$$

if

$$z = c^2 + b^2,$$

then

$$x = c^2 - b^2,$$

and

$$y = 2cb.$$

In $Z_3, z^2 \in \bar{1}_3$, because $\bar{2}_3$ contains no even powers and $\bar{0}_3$ integers do not equate to a sum of squares [4, 7] and $3 \nmid z$ [5]. Thus the Z_3 pPt Class function is:

$$\bar{1}_3 + \bar{0}_3 = \bar{1}_3. \quad (3.2)$$

Furthermore, the rows of z^2 are multiples of elements of the sequence of pentagonal numbers [4, 5]. For example, in Z_4 all odd squares where $3 \nmid N^2$ have rows equal to $6K$ where

$$K = \frac{1}{2}n(3n \pm 1), \quad (3.3)$$

the pentagonal numbers.

In $\bar{1}_3$, these rows r_1 are equivalent to $\frac{4}{3}R_1 = 8K$, so that integers formed with $(x^2)^*$, $(y^2)^*$ equal to (1,4), (5,6) or (5,0) must have rows equal to $4n(3n \pm 1)$ (Tables, 7,8).

Another constraint is that x, y cannot have a common factor [4, 7]. While this can lead to a triple it is not primitive. For example, for $(x^2, y^2)^* = (1,4)$, we have the triple (1225, 784, 441) which has 7 as a common factor, and which reduces to (25, 16, 9) for the squares:

$$\begin{aligned} 1225 &= 35^2 = (7 \times 5)^2, \\ 784 &= 28^2 = (7 \times 4)^2, \\ 441 &= 21^2 = (7 \times 3)^2. \end{aligned}$$

Note that the row of 1225 in $\bar{1}_3$, has the pattern $4n(3n - 1)$, $n = 6$:

$$408 = 4(102) = 4 \times 6(17) = 4 \times 6(3 \times 6) - 1).$$

N	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	4	2	14	3	30	4	52
5	80	6	114	7	154	8	200
9	252	10	310	11	374	12	444
13	520	14	602	15	690	16	784
17	884	18	990	19	1102	20	1220
21	1344	22	1430	23	1610	24	1752
25	1900	26	2054	27	2214	28	2380
29	2552	30	2730	31	2914	32	3104

...

33	3300	34	3502	35	3710	36	3924
37	4144	38	4370	39	4602	40	4840
41	5084	42	5334	43	5590	44	5852
45	6120	46	6394	47	6674	48	6960
49	7252	50	7550	51	7854	52	8164
53	8480	54	8802	55	9130	56	9464
57	9804	58	10150	59	10502	60	10860
61	11224	62	11594	63	11970	64	12352
65	12740	66	13134	67	13534	68	13940
69	14352	70	14770	71	15194	72	15624
73	16060	74	16502	75	16950	76	17404

Table 7: Pentagonal numbers $\times 2$: $f(n) = n(3n + 1)$

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	2	2	10	3	24	4	44
5	70	6	102	7	140	8	184
9	234	10	290	11	352	12	420
13	494	14	574	15	660	16	752
17	850	18	954	19	1064	20	1180
21	1302	22	1430	23	1564	24	1704
25	1850	26	2002	27	2160	28	2324
29	2494	30	2670	31	2852	32	3040
33	3234	34	3434	35	3640	36	3852
37	4070	38	4294	39	4524	40	4760
41	5002	42	5250	43	5504	44	5764
45	6030	46	6302	47	6580	48	6864
49	7154	50	7450	51	7752	52	8060
53	8374	54	8694	55	9020	56	9352
57	9690	58	10034	59	10389	60	10740
61	11102	62	11470	63	11844	64	12224
65	12610	66	13002	67	13400	68	13804
69	14214	70	14630	71	15052	72	15480
73	15914	74	16354	75	16800	76	17252

Table 8: Pentagonal numbers $\times 2$: $f(n) = n(3n - 1)$

4. Invalid RED Systems

(a) RED System (1,4)

In general,

$$z^2 = x^2 + y^2 \quad (4.1)$$

so that

$$3r_1 + 1 = 3r_1' + 1 + 3r_0 \quad (4.2)$$

and hence

$$r_1 = r_0 + r_1' \quad (4.3)$$

Since

$$(x^2)^* = 1, (r_1')^* \in \{0, 7\}, (y^2)^* = 4, (r_0)^* \in \{8, 1\} \Rightarrow (r_1)^* = 8.$$

(Note that each Z_3 class has even/odd integers.) Furthermore, since $3 \nmid z$ [6], the rows of z^2 follow the pentagonal numbers as noted above. Thus,

$$r_1 = 8\left(\frac{1}{2}n(3n \pm 1)\right) = 4n(3n \pm 1) \quad (4.4)$$

or

$$\frac{r_1}{4} = n(3n \pm 1). \quad (4.5)$$

Thus $(n(3n \pm 1))^*$ should equal 2 or 7, but $f(n)$ is always even, so we need only consider $(f(n))^* = (n(3n + 1))^* = 2$ (Tables 7,8). When $(f(n))^* = 2$ we get the second order linear inhomogeneous recurrence relation [2]

$$f(n)_{j+1} - 2f(n)_j + f(n)_{j-1} = 150, j \geq 1. \quad (4.6)$$

A method for solving such equations may be found in [3]. For instance, when $n = 24$, and $j = 4$, for $\{(f(n))^* = 2\}$:

$$f_5(24) - 2 \times f_4(24) + f_3(24) = 1752 - 2 \times 1102 + 602 = 150.$$

However, $(r_1/4)$ for z^2 with RED (1,4) for x^2, y^2 does not show any match with $(n(3n \pm 1))$ (Table 9). Any match only occurs when (x, y) pairs have a common factor so that the triple is not primitive. For a given x ,

$$f(n)_{j+1} - f(n)_j = \begin{cases} 160 & (\bar{0}_3, \bar{1}_3), \\ 180 & (\bar{1}_3, \bar{0}_3). \end{cases} \quad (4.7)$$

x^2	Class	y^2	Class	$f(n)$
1	$\bar{1}_3$	144-2304	$\bar{0}_3$	12,27,147,192
81	$\bar{0}_3$	4-4624	$\bar{1}_3$	7,47,72,127,232,287,327,392
121	$\bar{1}_3$	144-2304	$\bar{0}_3$	22,37,157,202
361	$\bar{1}_3$	144-2304	$\bar{0}_3$	57,177,222
441	$\bar{0}_3$	4-4624	$\bar{1}_3$	37,42,77,122,157,262,317,357,422*
841	$\bar{1}_3$	144-2304	$\bar{0}_3$	82,97,217,262
961	$\bar{1}_3$	144-2304	$\bar{0}_3$	92,107,217,227,272
1521	$\bar{0}_3$	4-4624	$\bar{1}_3$	127,132,187,192,212,247,352,407,447,512
1681	$\bar{1}_3$	144-2304	$\bar{0}_3$	152,187,287,332
2401	$\bar{1}_3$	144-2304	$\bar{0}_3$	212,227,347,392
2601	$\bar{0}_3$	4-3844	$\bar{1}_3$	217,222,257,282,302,442,497,537**
3481	$\bar{1}_3$	144-2304	$\bar{0}_3$	302,317,437,482
3721	$\bar{1}_3$	144-2304	$\bar{0}_3$	322,337,457,502
4761	$\bar{0}_3$	4-4624	$\bar{1}_3$	397,422,437,462,482,517,622,677,717,782

Table 9: $\{f(n)\}$: RED system (1,4)

[* $y = 784$ invalid as x, y have common factor; ** $y = 4624$ invalid as common factor 17]

(b) RED System (5,6)

Equation (4.3) holds again, but since

$$(x^2)^* = 5, (r_1')^* \in \{8,5\}, (y^2)^* = 6, (r_0)^* \in \{2,5\} \Rightarrow (r_1)^* = 0.$$

Thus $(r_1/4) = 0$ and 5 is invalid. For $(f(n))^* = 0$, both $n(3n+1)$ and $n(3n-1)$ satisfy the fourth order recurrence relation with suitable initial conditions (Table 10).

$$f(n)_{j+2} - 2f(n)_j + f(n)_{j-2} = 150, j \geq 2.$$

For instance, when $n = 25$, and $j = 7$, for $\{(f(n))^* = 0\}$:

$$f_9(25) - 2 \times f_7(25) + f_5(25) = 1430 - 2 \times 850 + 420 = 150. :$$

$f(n)$	$f(n)_0$	$f(n)_1$
$n(3n+1)$	30	80
$n(3n-1)$	10	70

Table 10: Initial values for $\{f(n)\}$

x^2	Class	y^2	Class	$f(n)$
25	$\bar{1}_3$	36-4536	$\bar{0}_3$	5,50,110,245,385
225	$\bar{0}_3$	16-4096	$\bar{1}_3$	20,35,40,75,115,180,195,280,360
625	$\bar{1}_3$	36-4536	$\bar{0}_3$	55,100,160,295,415
1225	$\bar{1}_3$	36-4536	$\bar{0}_3$	105,150,210,345,465
2025	$\bar{0}_3$	16-4096	$\bar{1}_3$	170,185,190,225,265,330,345,430,510
3025	$\bar{1}_3$	36-4536	$\bar{0}_3$	255,300,360,495,615
4225	$\bar{1}_3$	36-4536	$\bar{0}_3$	355,400,460,595,715

Table 11: $\{f(n)\}$: RED system (5,6)

(c) RED System (5,0)

Since x^2, y^2 must have a common factor, 5, no pPts can be formed from this system.

5. Final Comments

The formation of pPts has a number of constraints. The four major one are:

- $3 \nmid z$: this severely limits the possible values of z ;
- $z = c^2 + b^2$ so that factors equal to $(4r_3 + 3, \bar{3}_4)$ are invalid. If these factors are present in z , then a pPt cannot form, because, if $N \in \bar{3}_4$, then N is not equal to a sum of squares;
- The rows of the odd squares follow the pattern of the pentagonal numbers, or the triangular numbers when $3|x$ or y . The rows of z must therefore follow the pentagonal numbers;
- x and y cannot have a common factor.

These four constraints underlie the failure of REDs (1,4), (5,6) and (5,0) for the squares of x and y to be associated with pPts. Modular rings and their associated integer structure analysis are probably the simplest way to illustrate the above.

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