Note on the matrix Fermat’s equation
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Abstract. We consider the Fermat’s equation
\[ X^n + Y^n = Z^n \]
(F)
in the set of 2 × 2 rational matrices. We give some necessary condition of solvability of this equation.

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1 Introduction
The Fermat’s equation (F) in \( M_2(\mathbb{Q}) \) was considered by Barnett and Weitkamp in 1961 what was described by P. Ribenboim in monograph [13]. In 1966 R. Z. Domiaty [4] discovered that the equation (F) has infinitely many solutions in \( M_2(\mathbb{Z}) \) for \( n = 4 \). The solvability of (F) in \( GL_2(\mathbb{Z}) \) was first investigated by L. N. Vaserstein [14]. A. Khazanov in [9] gave necessary and sufficient conditions for solvability (F) for \( X, Y, Z \) belonging to \( SL_2(\mathbb{Z}), SL_3(\mathbb{Z}), GL_3(\mathbb{Z}) \). A. Grytczuk [7] proved some necessary condition to satisfy (F) in integral 2 × 2 matrices \( X, Y, Z \), and in [5] he gave an extension of this result. Studies connected with Khazanov’s results effected too H. Qin [12]. The equation of Fermat was investigated by Z. Patay and A. Szakas ([11]), Z. Cao and A. Grytczuk ([2]). In [3] Z. Cao and A. Grytczuk gave a necessary and sufficient condition for solvability (F) for \( X, Y, Z \in SL_2(\mathbb{Z}) \).

For \( X = A^x, Y = A^y, Z = A^z \) we obtain from (F) the following equation
\[ A^{nx} + A^{ny} = A^{nz} \]  
(1)
The necessary and sufficient conditions for solvability of the equation (1) in natural number \( x, y, z \) and \( n > 2 \), where \( A \in M_2(\mathbb{Z}) \) were given by Le and Li in [10]. Another proof of this result was given by A. Grytczuk ([6]). In the paper [8] we considered the extension of the equation (1)

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\[ A^{mx} + A^{my} + A^{mz} = A^{mw}, \]  
\[ \text{where } A \in M_2(\mathbb{Z}). \]

We gave ([8]) the necessary and sufficient conditions for solvability of the equation (2) in natural number \( x, y, z, w \) and \( n > 2 \).

In this paper we give the following necessary condition for solvability of the Fermat’s equation (F) in the set of 2 × 2 rational matrices \( M_2(\mathbb{Q}) \):

**Theorem 1** Let \( X, Y, Z \in M_2(\mathbb{Q}) \).

Let \( \det X = \det Y = k_1, \det Z = k_2 \), where \( k_1 \neq k_2, k_1, k_2 \neq 0 \), and

\[ Z^{-1}X = XZ^{-1}, Z^{-1}Y = YZ^{-1}, \]

\[ \text{Tr}Z^{-1}X, \text{Tr}Z^{-1}Y \in \{0, 2\}. \]

If the Fermat’s equation

\[ X^n + Y^n = Z^n \]  
(F)

has a solution, then \( n = 2 \) with

\[ \frac{k_1}{k_2} = -\frac{1}{2}. \]

In the proof of Theorem 1 we use the following lemma which can be easily prove by induction:

**Lemma 1** If \( A \in M_2(\mathbb{Q}) \), \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, b, d, f, h \neq 0 \), then

\[ A^n = \begin{pmatrix} F \left( \frac{a}{b} \right) & F \left( \frac{c}{d} \right) \\ F \left( \frac{f}{h} \right) & F \left( \frac{g}{h} \right) \end{pmatrix}, n \geq 2, \]

where \( w \) is a rational number, \( F \left( \frac{a}{b} \right) \), \( F \left( \frac{c}{d} \right) \) are polynomials of degree \( n \),

\[ F \left( \frac{a}{b} \right) - F \left( \frac{g}{h} \right) = \left( \frac{a}{b} - \frac{c}{d} \right) w. \]

Lemma 1 for \( A \in M_2(\mathbb{Z}) \) was proved in by K. Biala and A. Grytczuk in [1].

## 2 The proof of Theorem 1.

Let the assumptions of the Theorem 1 be satisfied.

From the equation (F) we obtain

\[ (Z^{-1}X)^n + (Z^{-1}Y)^n = I. \]  
(3)

Denote

\[ A = Z^{-1}X = XZ^{-1}, B = Z^{-1}Y = YZ^{-1}. \]

Hence, from (3) we have
\[ A^n + B^n = I. \] (4)

Let
\[
A = \begin{pmatrix} \frac{a_1}{b_1} & \frac{c_1}{d_1} \\ \frac{e_1}{f_1} & \frac{g_1}{h_1} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{a_2}{b_2} & \frac{c_2}{d_2} \\ \frac{e_2}{f_2} & \frac{g_2}{h_2} \end{pmatrix}.
\]

From Lemma 1 we obtain
\[
A^n = \begin{pmatrix} F_1\left(\frac{a_1}{b_1}\right) & \frac{c_1}{d_1}w_1 \\ \frac{e_1}{f_1}w_1 & F_1\left(\frac{g_1}{h_1}\right) \end{pmatrix}, \quad B^n = \begin{pmatrix} F_2\left(\frac{a_2}{b_2}\right) & \frac{c_2}{d_2}w_2 \\ \frac{e_2}{f_2}w_2 & F_2\left(\frac{g_2}{h_2}\right) \end{pmatrix},
\] (5)

where \(w_1, w_2\) are rational numbers,

\[
F_1\left(\frac{a_1}{b_1}\right) - F_1\left(\frac{g_1}{h_1}\right) = \left(\frac{a_1}{b_1} - \frac{e_1}{f_1}\right)w_1, \\
F_2\left(\frac{a_2}{b_2}\right) - F_2\left(\frac{g_2}{h_2}\right) = \left(\frac{a_2}{b_2} - \frac{e_2}{f_2}\right)w_2.
\]

By (4) and (5) it follows that
\[
\begin{pmatrix} F_1\left(\frac{a_1}{b_1}\right) & \frac{c_1}{d_1}w_1 \\ \frac{e_1}{f_1}w_1 & F_1\left(\frac{g_1}{h_1}\right) \end{pmatrix} + \begin{pmatrix} F_2\left(\frac{a_2}{b_2}\right) & \frac{c_2}{d_2}w_2 \\ \frac{e_2}{f_2}w_2 & F_2\left(\frac{g_2}{h_2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (6)

From (6) we give
\[
F_1\left(\frac{a_1}{b_1}\right) + F_2\left(\frac{a_2}{b_2}\right) = 1, \\
F_1\left(\frac{g_1}{h_1}\right) + F_2\left(\frac{g_2}{h_2}\right) = 1,
\] (7)

\[
\frac{c_1}{d_1}w_1 + \frac{c_2}{d_2}w_2 = \frac{e_1}{f_1}w_1 + \frac{e_2}{f_2}w_2 = 0.
\]

From the known theorem of Cauchy we have
\[
\det A^n = \det B^n = \left(\frac{k_1}{k_2}\right)^n.
\] (8)

From the other hand from (5) we obtain
\[
\det A^n = F_1\left(\frac{a_1}{b_1}\right) F_1\left(\frac{g_1}{h_1}\right) - \frac{c_1 e_1}{d_1 f_1}w_1^2, \\
\det B^n = F_2\left(\frac{a_2}{b_2}\right) F_2\left(\frac{g_2}{h_2}\right) - \frac{c_2 e_2}{d_2 f_2}w_2^2.
\] (9)

By (8), (9) and the last equation in (7) it follows that
\[ F_1 \left( \frac{a_1}{b_1} \right) F_1 \left( \frac{g_1}{h_1} \right) = F_2 \left( \frac{a_2}{b_2} \right) F_2 \left( \frac{g_2}{h_2} \right). \] (10)

From (5) we have
\[ TrA^n = F_1 \left( \frac{a_1}{b_1} \right) + F_1 \left( \frac{g_1}{h_1} \right), \] (11)
\[ TrB^n = F_2 \left( \frac{a_2}{b_2} \right) + F_2 \left( \frac{g_2}{h_2} \right). \] (12)

From (7) we give
\[ F_2 \left( \frac{g_2}{h_2} \right) = 1 - F_1 \left( \frac{g_1}{h_1} \right), \] (13)
\[ F_2 \left( \frac{a_2}{b_2} \right) = 1 - F_1 \left( \frac{a_1}{b_1} \right). \] (14)

From (10) and (13) we obtain
\[ \left( F_1 \left( \frac{a_1}{b_1} \right) + F_2 \left( \frac{a_2}{b_2} \right) \right) F_1 \left( \frac{g_1}{h_1} \right) = F_2 \left( \frac{a_2}{b_2} \right) \] (15)

By (15) and (7) it follows that
\[ F_1 \left( \frac{g_1}{h_1} \right) = F_2 \left( \frac{a_2}{b_2} \right) \] (16)

Similary we can prove that
\[ F_1 \left( \frac{a_1}{b_1} \right) = F_2 \left( \frac{g_2}{h_2} \right) \] (17)

Therefore from (7), (16), (17), (11) and (12) we obtain
\[ TrA^n = F_1 \left( \frac{a_1}{b_1} \right) + F_2 \left( \frac{a_2}{b_2} \right) = 1, \] (18)
\[ TrB^n = F_1 \left( \frac{g_1}{h_1} \right) + F_2 \left( \frac{g_2}{h_2} \right) = 1. \]

Let \( \lambda_1, \lambda_2 \) be the eigenvalues of the matrix A. Then \( \lambda_1^n, \lambda_2^n \) are the eigenvalues of the matrix \( A^n \).

From (18) and (8) we give
\[ TrA^n = \lambda_1^n + \lambda_2^n = 1, \] (19)
\[ \det A^n = \lambda_1^n \lambda_2^n = \left( \frac{k_1}{k_2} \right)^n. \] (20)

Let \( f(\lambda) = \lambda^2 - (TrA)\lambda + \det A \) be the characteristic polynomial of the matrix A.
Then
\[
\lambda_1 = \frac{TrA + \sqrt{(TrA)^2 - 4 \det A}}{2},
\]
\[
\lambda_2 = \frac{TrA - \sqrt{(TrA)^2 - 4 \det A}}{2},
\]
(21)
are the characteristic roots of A.

We consider the following cases:

1° \( TrA = 0 \)

Then from (21) we obtain
\[
\lambda_1 = \sqrt{-\frac{k_1}{k_2}}, \quad \lambda_2 = -\sqrt{-\frac{k_1}{k_2}}.
\]
(22)
For \( n = 2k + 1 \) from (22) we obtain
\[
TrA^n = 0
\]
what is contrary with (19).

For \( n = 2k \) from (19) and (22) we obtain
\[
TrA^n = 2 \left( \sqrt{-\frac{k_1}{k_2}} \right)^n = 1,
\]
thus
\[
\left( \sqrt{-\frac{k_1}{k_2}} \right)^n = \frac{1}{2}
\]
(23)
what is contrary for \( k_1, k_2 > 0 \) or \( k_1, k_2 < 0 \).

If \( k_1 < 0 \) and \( k_2 > 0 \) or \( k_2 < 0 \) and \( k_1 > 0 \), then
\[
-\frac{k_1}{k_2} > 0.
\]
Then the equation (23) is true for \( n = 2 \) and \( \frac{k_1}{k_2} = -\frac{1}{2} \).

For \( n = 2k \), where \( k_1 < 0 \) and \( k_2 > 0 \) or \( k_2 < 0 \) and \( k_1 > 0 \) from (22) we have
\[
\det A^n = (-1)^n \left( \frac{k_1}{k_2} \right)^n = \left( \frac{k_1}{k_2} \right)^n.
\]
Therefore (20) and (19) are satisfied for \( n = 2 \) with \( \frac{k_1}{k_2} = -\frac{1}{2} \).

2° \( TrA = 2 \)
Then from (21) we get

$$\lambda_1 = 1 + \sqrt{1 - \frac{k_1}{k_2}}, \quad \lambda_2 = 1 - \sqrt{1 - \frac{k_1}{k_2}}.$$  

We have

$$TrA^n = \left(1 + \sqrt{1 - \frac{k_1}{k_2}}\right)^n + \left(1 - \sqrt{1 - \frac{k_1}{k_2}}\right)^n$$

$$= 2 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^2 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^4 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^6 + ... +$$

$$+ \left(\sqrt{1 - \frac{k_1}{k_2}}\right)^n + \left(-\sqrt{1 - \frac{k_1}{k_2}}\right)^n \quad (24)$$

$$= 2 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^2 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^4 + 2\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^6 + ... +$$

$$+ \left(\sqrt{1 - \frac{k_1}{k_2}}\right)^n + \left(-\sqrt{1 - \frac{k_1}{k_2}}\right)^n.$$  

Let $n = 2k$.

Then we have

$$\left(\sqrt{1 - \frac{k_1}{k_2}}\right)^n + \left(-\sqrt{1 - \frac{k_1}{k_2}}\right)^n = 2\left(1 - \frac{k_1}{k_2}\right)^n. \quad (25)$$

From (24), (25) and (19) we obtain

$$1 + \left(\frac{n}{2}\right)\left(1 - \frac{k_1}{k_2}\right) + ... + \left(\frac{n}{n - 2}\right)\left(1 - \frac{k_1}{k_2}\right)\frac{1}{2^{(n-2)}} + \left(1 - \frac{k_1}{k_2}\right)^k = \frac{1}{2}. \quad (26)$$

Assume that $\frac{k_1}{k_2} < 1$.

Then

$$1 + \left(\frac{n}{2}\right)\left(1 - \frac{k_1}{k_2}\right) + ... + \left(\frac{n}{n - 2}\right)\left(1 - \frac{k_1}{k_2}\right)\frac{1}{2^{(n-2)}} + \left(1 - \frac{k_1}{k_2}\right)^k > 1 > \frac{1}{2},$$

therefore the equation (26) does not hold.

Assume that $\frac{k_1}{k_2} > 1$.

Then we remark that (26) is not satisfied.

Let $n = 2k + 1$.  

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Then
\[
\left(\sqrt{1-k_1/k_2}\right)^n + \left(-\sqrt{1-k_1/k_2}\right)^n = 0. \tag{27}
\]

From (24), (27) and (19) we have
\[
1 + \left(\frac{n}{2}\right)(1-k_1/k_2) + \left(\frac{n}{4}\right)(1-k_1/k_2)^2 + \ldots + \left(\frac{n}{2k}\right)(1-k_1/k_2)^k = \frac{1}{2}. \tag{28}
\]

If \(0 < k_1/k_2 < 1\), then
\[
1 + \left(\frac{n}{2}\right)(1-k_1/k_2) + \left(\frac{n}{4}\right)(1-k_1/k_2)^2 + \ldots + \left(\frac{n}{2k}\right)(1-k_1/k_2)^k > \frac{1}{2}
\]
and the equation (28) is not satisfied.

If \(k_1/k_2 > 1\), then
\[
1 + \left(\frac{n}{2}\right)(1-k_1/k_2) + \left(\frac{n}{4}\right)(1-k_1/k_2)^2 + \ldots + \left(\frac{n}{2k}\right)(1-k_1/k_2)^k \neq \frac{1}{2}.
\]

Similarly as for the matrix A we obtain for the matrix B and the proof of Theorem 1 is finished. ■

**Example 1** Let \(X = Y = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\)

Then
\[
Z^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \det X = \det Y = -\frac{1}{2}, \ \det Z = 1,
\]
\[
Z^{-1}X = XZ^{-1}, Z^{-1}Y = YZ^{-1},
\]
\[
\det A = \det XZ^{-1} = -\frac{1}{2}, \ \text{Tr} A = 0.
\]

We have \(X^2 + Y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Z^2.\)

**References**


