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On multiplicative functions with strictly positive values

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Abstract

The paper is a continuation of [1] and [2]. The considerations are over the class of multiplicative functions with strictly positive values and more precisely, over the pairs (f, g) of such functions, which have a special property, called in the paper property $\mathbb{S}$. For every two such pairs (f_1, g) and (f_2, g), with different f_1 and f_2, a sufficient **condition** for the coincidence of the maximum (respectively of the minimum) of the numbers $f_1(d)g\left(\frac{n}{d}\right)$ and $f_2(d)g\left(\frac{n}{d}\right)$, where d runs over all proper divisors of an arbitrary composite number $n > 1$, is given. Some applications of the results are made for several classical multiplicative functions like Euler's totient function φ, Dedekind's function ψ, the sum of all divisors of m, i.e. $\sigma(m)$, the number of all divisors of m, i.e. $\tau(m)$, and $2^{\omega(m)}$, where $\omega(m)$ is the number of all prime divisors of m.

Keywords: multiplicative functions, divisors, proper divisors, prime numbers, composite number

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Used Denotations: $\mathbb{Z}^+$ - the set of all non-negative integers; $\mathbb{N}$ - the set of all positive integers; $\mathbb{P}$ - the set of all prime numbers; for a given $n \in \mathbb{N}$ D_n^* denotes the set of all proper divisors of n, i.e. different than 1 and n; for $n > 1$ $\omega(n)$ denotes the number of all prime divisors of n ($\omega(1) \stackrel{\text{def}}{=} 0$); for $a, b \in \mathbb{N}$ $\gcd(a, b)$ denotes the greatest common divisor of a and b; for $p \in \mathbb{P}$ $\text{ord}_p n$ denotes the largest exponent k for which p^k is a divisor of n.

1 Introduction

The present paper is a continuation of the research from [1] and [2]. We remind that an artihmetic function F is said to be multiplicative if for every $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$ it is fulfilled

$$F(ab) = F(a)F(b)$$

Therefore, $F(1) = 1$ if $F \not\equiv 0$.

Some classical examples of multiplicative functions that have an importnat meaning in Number Theory are Euler's totient function (the function φ), Dedekind's function (the function ψ), sum of all divisors of a positive integer (the function σ) and the number of all divisors of a positive integer (the function τ). When $n > 1$ these functions admit the following multiplicative representations:

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right);$$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right);$$

$$\sigma(n) = \prod_{p|n} \frac{p^{1+\mathrm{ord}_p n} - 1}{p - 1};$$

$$\tau(n) = \prod_{p|n} \left(1 + \mathrm{ord}_p n\right),$$

where p runs over all prime divisors of n.

Another example of classical multiplicative function is $2^{\omega(m)}$. For all of the above multiplicative functions see [3] and [4, p. 20, p. 33, p. 39, p. 180, p. 231, p. 284].

2 Main results

Below we shall consider only the class $\mathbb{M}$ of all multiplicative functions with strictly positive values. Our investigation is based on some pairs of multiplicative functions from the class $\mathbb{M}$ which have a special property (called in the paper property $\mathbb{S}$). For such pairs in [1] and [2] the question about finding the $\max\limits_{d \in \mathbb{D}_n^*} \left\{f(d)g\left(\frac{n}{d}\right)\right\}$ and $\min\limits_{d \in \mathbb{D}_n^*} \left\{f(d)g\left(\frac{n}{d}\right)\right\}$, when $n > 1$ is a composite number, is completely solved. Since some pairs of classical multiplicative functions with strictly positive values (like (φ, σ), (φ, ψ), (τ, σ),$(2^{\omega(m)}, \sigma)$, $(2^{\omega(m)}, \psi)$, $(2^{\tau}, \psi)$)have property $\mathbb{S}$, we will apply our results to them obtaining as a corollary some new theorems.

The main results of the paper are Theorem 3, Theorem 4 and Corollaries 1-4.

Definition. *Let $f, g \in \mathbb{M}$. We say that the ordered pair (f, g) has the property $\mathbb{S}$ when one of the following two cases is fulfilled:*

(i) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$

$$H_{p,m}^{f,g}(k) \stackrel{\text{def}}{=} f(p^k)g(p^{m-k}) \tag{1}$$

is an increasing function (not necessarily strictly) with respect to $k \in [0, m] \cap \mathbb{Z}^+$

(ii) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$ *the function* $H_{p,m}^{f,g}$ *from* (1) *is a decreasing function (not necessarily strictly) with respect to* $k \in [0, m] \cap \mathbb{Z}^+$

Our investigation is based on the following two theorems which are contained in [1, Theorem 5]:

Theorem 1. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)). If $n > 1$ is a composite number, then:*

$$\max_{d\in\mathbb{D}_n^*}\left\{f(d)g\left(\frac{n}{d}\right)\right\} = \max_{p}\left\{g(p)f\left(\frac{n}{p}\right)\right\} \tag{2}$$

$$\min_{d\in\mathbb{D}_n^*}\left\{f(d)g\left(\frac{n}{d}\right)\right\} = \min_{p}\left\{f(p)g\left(\frac{n}{p}\right)\right\}, \tag{3}$$

where p runs over all prime divisors of n.

Theorem 2. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (ii)). If $n > 1$ is a composite number, then:*

$$\max_{d\in\mathbb{D}_n^*}\left\{f(d)g\left(\frac{n}{d}\right)\right\} = \max_{p}\left\{f(p)g\left(\frac{n}{p}\right)\right\} \tag{4}$$

$$\min_{d\in\mathbb{D}_n^*}\left\{f(d)g\left(\frac{n}{d}\right)\right\} = \min_{p}\left\{g(p)f\left(\frac{n}{p}\right)\right\} \tag{5}$$

where p runs over all prime divisors of n.

Theorem 1 and Theorem 2 provide some important corollaries. Namely let $f_j \in \mathbb{M}, j = \overline{1,2}$ be two different functions such that they satisfy the **condition**:

$$f_1(p) = f_2(p)\ \forall p \in \mathbb{P}$$

In this case from Theorem 1 ((3)) as a corollary we obtain:

Theorem 3. *Let $g \in \mathbb{M}$ and the pairs $(f_j, g), j = \overline{1,2}$ have the property $\mathbb{S}$ (satisfying (i)). Then for every composite number $n > 1$*

$$\min_{d\in\mathbb{D}_n^*}\left\{f_1(d)g\left(\frac{n}{d}\right)\right\} = \min_{d\in\mathbb{D}_n^*}\left\{f_2(d)g\left(\frac{n}{d}\right)\right\} \tag{6}$$

Also from Theorem 2 ((4)) as a corollary we obtain:

Theorem 4. *Let $g \in \mathbb{M}$ and the pairs $(f_j, g), j = \overline{1,2}$ have the property $\mathbb{S}$ (satisfying (ii)). Then for every composite number $n > 1$*

$$\max_{d\in\mathbb{D}_n^*}\left\{f_1(d)g\left(\frac{n}{d}\right)\right\} = \max_{d\in\mathbb{D}_n^*}\left\{f_2(d)g\left(\frac{n}{d}\right)\right\} \tag{7}$$

Since when d runs over all proper divisors of n, $\frac{n}{d}$ runs over the same divisors too and having in mind the following

Remark. *If $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)), then the pair (g, f) has the property $\mathbb{S}$ (satisfying (ii)).*

it is clear that other variants of such theorems (like Theorem 3 and Theorem 4) cannot be deduced from Theorem 1 and Theorem 2.

2.1 Application of the results

First we shall make an application of Theorem 3. Putting $f_1 = \sigma, f_2 = \psi$ we have

$$f_1(p) = f_2(p)\ \forall p \in \mathbb{P}$$

since

$$\sigma(p) = \psi(p) = p + 1.$$

Now, let $g = \varphi$. In this case it is easy to verify that the pairs (σ, φ) and (ψ, φ) have the property $\mathbb{S}$ (satisfying (i)). Therefore, we obtain from Theorem 3:

Corollary 1. *Let $n > 1$ be a composite number. Then*

$$\min_{d \in \mathbb{D}_n^*} \left\{ \sigma(d)\varphi\left(\frac{n}{d}\right) \right\} = \min_{d \in \mathbb{D}_n^*} \left\{ \psi(d)\varphi\left(\frac{n}{d}\right) \right\}$$

In the same manner, putting $g = \tau$ and observing that the pairs (σ, τ) and (ψ, τ) have the property $\mathbb{S}$ (satisfying (i)), from Theorem 3 we obtain:

Corollary 2. *Let $n > 1$ be a composite number. Then*

$$\min_{d \in \mathbb{D}_n^*} \left\{ \sigma(d)\tau\left(\frac{n}{d}\right) \right\} = \min_{d \in \mathbb{D}_n^*} \left\{ \psi(d)\tau\left(\frac{n}{d}\right) \right\}$$

Second we shall make an application of Theorem 4. Putting $f_1 = 2^{\omega(m)}, f_2 = \tau$ we have

$$f_1(p) = f_2(p)\ \forall p \in \mathbb{P}$$

since

$$2^{\omega(p)} = \tau(p) = 2.$$

Now, let $g = \sigma$. In this case it is easy to verify that the pairs $(2^{\omega(m)}, \sigma)$ and (τ, σ) have the property $\mathbb{S}$ (satisfying (ii)). Therefore, we obtain from Theorem 4:

Corollary 3. *Let $n > 1$ be a composite number. Then*

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)}\sigma\left(\frac{n}{d}\right) \right\} = \max_{d \in \mathbb{D}_n^*} \left\{ \tau(d)\sigma\left(\frac{n}{d}\right) \right\}$$

In the same manner, putting $g = \psi$ and observing that the pairs $(2^{\omega(m)}, \psi)$ and (τ, ψ) have the property $\mathbb{S}$ (satisfying (ii)), from Theorem 4 we obtain:

Corollary 4. *Let $n > 1$ be a composite number. Then*

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)}\psi\left(\frac{n}{d}\right) \right\} = \max_{d \in \mathbb{D}_n^*} \left\{ \tau(d)\psi\left(\frac{n}{d}\right) \right\}$$

Finally, we make some observations concerning unitary divisors of a number.

Let $n \in \mathbb{N}$ be arbitrary number. We remind that a divisor d of n is said to be unitary divisor of n if $\gcd(d, \frac{n}{d}) = 1$. A unitary divisor d of n, such that $d \neq 1, n$ is said to be proper unitary divisor of n.

Let Ξ_n be an arbitray set whose elements are unitary divisors of n (not necessarily all) and having the property

$$\text{If } d \in \Xi_n \text{ then } \frac{n}{d} \in \Xi_n$$

In particular, the set of all unitary divisors of n and the set of all proper unitary divisors of n are examples of such sets Ξ_n. Let $f, g \in \mathbb{M}$. From [1, Lemma 1] it is trivial to obtain the following result.

Proposition 1. *Let*

$$\max_{d \in \Xi_n} \left\{ f(d) g\left(\frac{n}{d}\right) \right\}$$

is reached for $d = d'$*, then*

$$\min_{d \in \Xi_n} \left\{ f(d) g\left(\frac{n}{d}\right) \right\}$$

is reached for $\frac{n}{d'}$*. If*

$$\min_{d \in \Xi_n} \left\{ f(d) g\left(\frac{n}{d}\right) \right\}$$

is reached for d''*, then*

$$\max_{d \in \Xi_n} \left\{ f(d) g\left(\frac{n}{d}\right) \right\}$$

is reached for $\frac{n}{d''}$*.*

For the particular case, $g = f$ we obtain the following result:

Proposition 2. *All elements of the set*

$$\left\{ f(d) f\left(\frac{n}{d}\right) : d \text{ runs over all elements of } \Xi_n \right\}$$

are equal and coinciding with $f(n)$*. If:*

$$\min_{d \in \Xi_n} \{f(d)\} = A_n; \ \max_{d \in \Xi_n} \{f(d)\} = B_n,$$

then

$$A_n B_n = f(n)$$

Example 1. *Let* $n = 60$ *and* $f = \sigma$*. If* Ξ_{60} *is the set of all unitary divisors of* 60*, we have* $A_{60} = 1,\ B_{60} = 168$ *and* $A_{60} B_{60} = 1.168 = 168 = \sigma(60)$.

If Ξ_{60} *is the set of all proper unitary divisors of* 60*, we have* $A_{60} = 4,\ B_{60} = 42$ *and* $A_{60} B_{60} = 4.42 = 168 = \sigma(60)$

It is easy to see that if $n > 1$ is a composite squarefree number, then all divisors of n are unitary divisors of n and all proper divisors of n are proper unitary divisors of n. Therefore, we may choose for Ξ_n each one of the sets: the set of all divisors of n or the set of all proper divisors of n and Proposition 2 remains valid for such n.

Example 2. *Let* $n = 105$ *and* $f = \varphi$*. If* Ξ_{105} *is the set of all divisors of* 105*, we have* $A_{105} = 1,\ B_{105} = 48$ *and* $A_{105} B_{105} = 1.48 = 48 = \varphi(105)$.

If Ξ_{105} *is the set of all proper divisors of* 105*, we have* $A_{105} = 2,\ B_{105} = 24$ *and* $A_{105} B_{105} = 2.24 = 48 = \varphi(105)$.

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