A note on switching in symmetric $n$-sigraphs

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Abstract: In this note, we define switching in a different manner and obtained some results on symmetric $n$-sigraphs.

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1 Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An $n$-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric $n$-tuples. Note that $H_n$ is a group under coordinate wise multiplication, and the order of $H_n$ is $2^m$, where $m = \left\lceil \frac{n}{2} \right\rceil$.

A symmetric $n$-sigraph (symmetric $n$-marked graph) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the underlying graph of $S_n$ and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function.

In this paper by an $n$-tuple/n-sigraph/n-marked graph we always mean a symmetric $n$-tuple/symmetric $n$-sigraph/symmetric $n$-marked graph.

An $n$-tuple $(a_1, a_2, ..., a_n)$ is the identity $n$-tuple, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a non-identity $n$-tuple. In an $n$-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity $n$-tuple is called an identity edge, otherwise it is a non-identity edge.

Further, in an $n$-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(S_n)$ the $n$-tuple $\sigma(A)$ is the product of the $n$-tuples on the edges of $A$.

In [7], the authors defined two notions of balance in $n$-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. Siva Kota Reddy [4]):

Definition. Let $S_n = (G, \sigma)$ be an $n$-sigraph. Then, (i) $S_n$ is identity balanced (or $i$-balanced), if product of $n$-tuples on each cycle of $S_n$ is the identity $n$-tuple, and (ii) $S_n$ is balanced, if every cycle in $S_n$ contains an even number of non-identity edges.

Note: An $i$-balanced $n$-sigraph need not be balanced and conversely. The following characterization of $i$-balanced $n$-sigraphs is obtained in [7].

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Proposition 1. (E. Sampathkumar et al. [7]) An n-sigraph \( S_n = (G, \sigma) \) is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge \( uv \) is equal to the product of the n-tuples of \( u \) and \( v \).

In [7], the authors also have defined switching and cycle isomorphism of an n-sigraph \( S_n = (G, \sigma) \) as follows: (See also [5, 6] & [9]-[12]).

Let \( S_n = (G, \sigma) \) and \( S'_n = (G', \sigma') \), be two n-sigraphs. Then \( S_n \) and \( S'_n \) are said to be isomorphic, if there exists an isomorphism \( \phi : G \to G' \) such that if \( uv \) is an edge in \( S_n \) with label \((a_1, a_2, ..., a_n)\) then \( \phi(u)\phi(v) \) is an edge in \( S'_n \) with label \((a_1, a_2, ..., a_n)\).

Given an n-marking \( \mu \) of an n-sigraph \( S_n = (G, \sigma) \), switching \( S_n \) with respect to \( \mu \) is the operation of changing the n-tuple of every edge \( uv \) of \( S_n \) by \( \mu(u)\sigma(uv)\mu(v) \). The n-sigraph obtained in this way is denoted by \( \mathcal{S}_\mu(S_n) \) and is called the \( \mu \)-switched n-sigraph or just switched n-sigraph.

Further, an n-sigraph \( S_n \) switches to n-sigraph \( S'_n \) (or that they are switching equivalent to each other), written as \( S_n \sim S'_n \), whenever there exists an n-marking of \( S_n \) such that \( \mathcal{S}_\mu(S_n) \cong S'_n \).

Two n-sigraphs \( S_n = (G, \sigma) \) and \( S'_n = (G', \sigma') \) are said to be cycle isomorphic, if there exists an isomorphism \( \phi : G \to G' \) such that the n-tuple \( \sigma(C) \) of every cycle \( C \) in \( S_n \) equals to the n-tuple \( \sigma(\phi(C)) \) in \( S'_n \).

Proposition 2. (E. Sampathkumar et al. [7]) Given a graph \( G \), any two n-sigraphs with \( G \) as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

2 New version of switching

Let \( S_n = (G, \sigma) \) be an n-sigraph and \( v \in V(S) \). The n-sigraph \( S_n^v = (G, \sigma^v) \) is said to be obtained from \( S_n \) by switching \( v \). The n-tuple \( \sigma(A) \) is

\[
\sigma^v(e) = \begin{cases} 
-\sigma(e), & \text{if } v \text{ is an end point of } e \\
\sigma(e), & \text{otherwise.}
\end{cases}
\]

Proposition 3. Let \( S_n = (G, \sigma) \) be an n-sigraph and let \( u \) and \( v \) be two vertices of \( S_n \). Then \( (S_n^u)^v = (S_n^v)^u \).

Proof. If \( (\sigma^v)^u = (\sigma^u)^v \) then proof is over. There are three cases: \( e \) has neither \( u \) or \( v \) as a vertex, \( e \) has exactly one of \( u \) or \( v \) as a vertex, or \( e \) has both \( u \) and \( v \) as vertices. If \( e \) is incident to neither \( u \) nor \( v \) then \( (\sigma^v)^u(e) = (\sigma^u)^v(e) = (\sigma(uv)\sigma(e)) = (\sigma^u)^v(e) \). If \( e \) is incident to only one of \( u \) or \( v \), then without loss of generality we let \( e \) be incident to \( u \). Now we see that \( (\sigma^v)^u(e) = (\sigma^u)^v(e) = -\sigma(e) = -\sigma^v(e) = (\sigma^u)^v(e) \). Finally, if \( e \) is incident to both \( u \) and \( v \), then \( (\sigma^v)^u(e) = (\sigma^u)^v(e) = \sigma(e) = -\sigma^u(e) = (\sigma^u)^v(e) \).

For \( U \subseteq V(S_n) \), \( S_n^U \) is the n-sigraph obtained by switching each of the vertices of \( U \). By Proposition 3, the order in which the vertices are switched does not matter. An n-sigraph \( S'_n = (G', \sigma') \) is switching equivalent to \( S_n = (G, \sigma) \), if \( S'_n = (G', \sigma') \cong S_n^U \) for some \( U \subseteq V(S_n) \). The set of n-sigraphs switching equivalent to \( S_n = (G, \sigma) \) is called the switching class of \( S_n = (G, \sigma) \), written \([S_n]_s\).

For any \( a \in \{+, -, \} \), let \( \overline{a} \in \{+, -, \} \setminus \{a\} \). In an n-tuple \( (a_1, a_2, ..., a_n) \), the elements \( a_{\lceil \frac{n}{2} \rceil} \) and \( a_{\lceil \frac{n+1}{2} \rceil} \) are called middle elements. Note that an n-tuple has two middle elements if \( n \) is even and exactly one if \( n \) is odd. We now define various operations on an n-tuple \( (a_1, a_2, ..., a_n) \) as follows:
i) $f$-complement, $(a_1, a_2, ..., a_n)^f = (\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)$

ii) $m$-complement $(a_1, a_2, ..., a_n)^m = (b_1, b_2, ..., b_n)$ where,

$$b_k = \begin{cases} 
\bar{a}_k, & \text{if } a_k \text{ is a middle element;} \\
 a_k, & \text{Otherwise}
\end{cases}$$

iii) $e$-complement $(a_1, a_2, ..., a_n)^e = (b_1, b_2, ..., b_n)$ where,

$$b_k = \begin{cases} 
\bar{a}_k, & \text{if } a_k \text{ is not a middle element;} \\
 a_k, & \text{Otherwise}
\end{cases}$$

Let $t \in \{f, e, m\}$. Then $t$-complement $S^t_n$ of an $n$-sigraph $S_n = (G, \sigma)$ is obtained from $S_n$ by replacing each $n$-tuple on the edges of $S_n$ by its $t$-complement.

**Proposition 4.** Let $U \subseteq V(S_n)$. The graph obtained by $f$-complement of $n$-tuples on the edges in the cut $[U; U^f]$ is $S^U_n$.

**Proof.** The only way for an edge of $S^U_n$ to have a different $n$-tuple $(a_1, a_2, ..., a_n)$ than it had in $S_n$ is if it has exactly one endpoint in $U$. If it has none, its $n$-tuple $(a_1, a_2, ..., a_n)$ never changes. On the other hand if it has two endpoints in $U$ then it is $f$-complement twice (once for each endpoint in $U$). Thus the two $n$-sigraphs are the same. \qed

**Proposition 5.** If $S_n = (G, \sigma)$ is $i$-balanced then so is any switching of $S_n$.

**Proof.** Let $U \subseteq V(S_n)$. By Proposition 4, $S^U_n$ is obtained by $f$-complement of $n$-tuples on the edges in the cut $[U; U^f]$. The intersection of a cycle with a cut must always contain the number of $n$-tuples in any cycle whose $k^{th}$ co-ordinate is $-$ is even, and therefore $f$-complement those edges has no change on their product(it is again $i$-balanced). Thus the $n$-tuple of a cycle remains unchanged. \qed

**Proposition 6.** The switching class $[S_n]$ contains only identity $n$-tuples if, and only if, $S_n = (G, \sigma)$ is $i$-balanced.

**Remark.** In [1], the author introduced above switching for signed graphs. In this note, we generalized this switching for $n$-sigraphs. Further, the above new type switching defined in $n$-sigraphs is coincidence with switching already defined in $n$-sigraphs for there exists an $n$-marking of $n$-sigraphs.

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References


