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The structure of geometric number sequences

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Abstract: An integer structure analysis (ISA) of the triangular, tetrahedral, pentagonal and pyramidal numbers is developed. The relationships among the elements and the powers of the elements of these sequences are discussed. In particular, the triangular and pentagonal numbers are directly linked and are structurally important for the formation of triples. The class structure in the modular rings Z_3 and Z_4 of some elements of these sequences reinforce previous studies of their properties.

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1 Introduction

Sequences of integers obtained from geometric figures such as triangles, tetrahedra, pentagons and pyramids, have been known for centuries [2]. Here we analyse the some of the relationships among these sequences in the framework of integer structures within modular rings.

2 Geometric number sequences

(i) Triangular numbers $\{T_n\}$

These satisfy the third order linear homogeneous recurrence relation

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, n > 3,$$

with initial conditions 1, 3, 6, but they are more readily formed from the first order recurrence relation:

$$T_n = T_{n-1} + n$$

with initial value $T_1 = 1$, and solution

$$T_n = \tfrac{1}{2}n(n+1), n \in Z_+,$$

such that

$$\{T_n\} = \{1, 3, 6, 10, 15, 21, 28, 36.45.55, ...\},$$

some of which are both triangular and square [1]; for instance,

$$1, 36, 1225, 41616, 1413721, 4802490, ...$$

Triangular numbers are useful in Integer Structure Analysis (ISA) [3]. In modular rings such as Z_3 and Z_4 (Tables 1 and 2), the squares of odd integers N, with $3|N$, always fall in rows, R, given by

$$R = 3 + 24\sum \tfrac{1}{2} n(n+1), (Z_3), \tag{2.1}$$

and

$$R = 2 + 18\sum \tfrac{1}{2} n(n+1), (Z_4), \tag{2.2}$$

for non-negative integers n.

Row	f(r)	$3r_0$	$3r_1 + 1$	$3r_2 + 2$
	Class	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$
0		0	1	2
1		3	4	5
2		6	7	8
3		9	10	11
4		12	13	14
5		15	16	17
6		18	19	20
7		21	22	23

Table 1. Modular Ring Z_3

Row	$f(r)$	$4r_0$	$4r_1 + 1$	$4r_2 + 2$	$4r_3 + 3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 2. Rows of Z_4

The functions (2.1) and (2.2) only work for $3|N$ otherwise the odd N^2 always fall in rows containing elements of the sequence of pentagonal numbers and some of the triangular numbers with 3 as a factor are also pentagonal numbers (Section iii). This structural difference in the rows has important ramifications for power functions, such as triples, and the distribution of integers which are sums of squares [3, 6, 7].

Some properties of triangular numbers follow. The sum of consecutive triangular numbers are the squares of the integers; that is,

$$T_{n-1} + T_n = n^2. \tag{2.3}$$

Similarly, the sum of consecutive cubes yields the square of the triangular numbers:

$$\sum_{i=1}^{n} i^3 = T_n^2 \tag{2.4}$$

The most common factor of triangular numbers is 3, and when 3 is not a factor:

$$T_{n_{i+1}} - T_{n_i} = 9k, k = 1, 2, 3, ..., \tag{2.5}$$

And the corresponding n follow ($n = 1 + 3t$). Thus, $T_{n_{i+1}} - T_{n_i} = \frac{1}{2}\left(9t(t+1)\right)$ so $k = \frac{1}{2}t(t+1)$.
When 3 is a factor of T_n

$$\tfrac{1}{3}T_n = \tfrac{1}{2}n(3n \pm 1), \tag{2.6}$$

an interesting link with the pentagonal numbers as noted above.

(ii) Tetrahedral numbers {Hn}

These satisfy the fourth order linear homogeneous recurrence relation

$$H_n = 4H_{n-1} - 6H_{n-2} + 4H_{n-3} - H_{n-4},\ n > 4,$$

with initial conditions 1, 4, 10, 20, but they are more readily formed from the first order recurrence relation:

$$H_n = H_{n-1} + \frac{1}{2}n(n+1)$$

with initial value $H_1 = 1$, and solution

$$H_n = \binom{n+3}{3},\ n \in Z_+,$$

such that

$$\{H_n\} = \{1, 4, 10, 20, 35, 50, 84, 120, 165, 220, ...\},$$

and

$$H_n = \sum_{j=1}^{n} T_j \tag{2.7}$$

Ball piling and other forms of close packing are important in engineering and chemistry, and tetrahedral numbers in the geometric sense represent this [2].

(iii) Pentagonal numbers {Dn}

These satisfy the third order linear homogeneous recurrence relation

$$D_n = 3D_{n-1} - 3D_{n-2} + D_{n-3}, n > 3,$$

with initial conditions 1, 5, 12, but they are more readily formed from the first order recurrence relation:

$$D_n = D_{n-1} + 3n - 2$$

with initial value $D_1 = 1$, and solution

$$D_n = \tfrac{1}{2}n(3n-1),\ n \in Z_+, \tag{2.9}$$

such that

$$\{D_n\} = \{1, 5, 12, 22, 35, 51, 70, ...\},$$

with a companion pentagonal sequence (not mentioned in [9]):

$$\{C_n\} = \{2, 7, 15, 26, 40, 57, 77, ...\},$$

with general term

$$C_n = \tfrac{1}{2}n(3n+1),\ n \in Z_+, \tag{2.10}$$

so that

$$C_n = D_n + n\ .$$

Like the triangular numbers, the pentagonal numbers play a useful role in ISA. In the modular rings, Z_3 and Z_4, the square of odd integers N (with N not divisible by 3), fall in the rows, R, given by the rather neat inter-relationships [3]:

$$R = 4n(3n \pm 1) \in Z_3 \tag{2.11}$$

or

$$R = 3n(3n \pm 1) \in Z_4 \tag{2.12}$$

Since only integers not divisible by 3 have squares in these rows, this has an effect on the formation of Pythagorean triples [3, 8] and seems to be associated with the limitations on triples with powers greater than two [3, 6].

(iv) Pyramidal numbers {Qn}

These satisfy the fourth order linear homogeneous recurrence relation

$$Q_n = 4Q_{n-1} - 6Q_{n-2} + 4Q_{n-3} - Q_{n-4},\ n > 4,$$

with initial conditions 1, 5, 14, 30, but they are more readily formed from the first order recurrence relation:

$$Q_n = Q_{n-1} + n^2$$

with initial value $Q_1 = 1$, and solution

$$Q_n = \tfrac{1}{6} n(n+1)(2n+1),\ n \in Z_+,$$

such that

$$\{Q_n\} = \{1, 5, 14, 30, 55, 91, 140, 204, 285, 385, ...\},$$

and

$$Q_n - Q_{n-2} = n^2 + (n-1)^2 \tag{2.13}$$

which is why they are usually called square pyramidal numbers [10, #M3844]. As with tetrahedral numbers, in the geometric sense, pyramidal numbers represent normal piling.

3 Structure of these sequences in Z_3

The distribution of the various sequences over the three classes of Z_3 (Table 3) shows that no element of the triangular sequence belongs to Class $\bar{2}_3$. Since, if

$$\tfrac{1}{2} n(n+1) = 3r_2 + 2 \tag{3.1}$$

then

$$n = \tfrac{1}{2}\sqrt{-1(2(12r_2 + 9))} \tag{3.2}$$

The square root function of Equation (3.2) may be expressed as $4(6r_2 + 4) + 2$ which represents and integer in Class $\bar{2}_4 \in Z_4$ (Table 2). Since this class has no powers, then no n of Equation (3.2) is an integer.

- When 3 divides N, the integers all belong to Class $\bar{0}_3$.
- When 3 does not divide N, odd powers and primes belong to Classes $\bar{1}_3$, $\bar{2}_3$.
- When 3 does not divide N^m (m even), N^m belongs to Class $\bar{1}_3$; for example:

$$(3r_2 + 2)^2 = 3(3r_3^2 + 4r_3 + 1) + 1 \tag{3.3}$$

The class patterns are simple for the triangular and pentagonal numbers which are important for squares. On the other hand the tetrahedral and pyramidal sequences require nine class transfers to form their characteristic pattern. For the triangular numbers, the simplest sequence has a factor of 3 among a majority of the elements (Section 2 (i))) so that Class $\bar{0}_3$ is dominant.

Type	Class function for $n \in Z_3$			Class patterns
	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$	
Triangular	$n = a + 3t$			$\bar{1}_3\ \bar{0}_3\ \bar{0}_3$
	$a = 2, 3$	$a = 1$	nil	
Tetrahedral	$n = a + 9t$			$\bar{1}_3\ \bar{1}_3\ \bar{1}_3\ \bar{2}_3\ \bar{2}_3\ \bar{0}_3\ \bar{0}_3\ \bar{0}_3$
	$a = 8, 9$	$a = 1, 2, 3$	$a = 4, 5, 6$	
Pentagonal	$n = a + 3t$			$\bar{2}_3\ \bar{1}_3\ \bar{0}_3$
	$a = 0$	$a = 2$	$a = 1$	$\bar{1}_3\ \bar{2}_3\ \bar{0}_3$
	$a = 0$	$a = 1$	$a = 2$	
Pyramidal	$n = a + 9t$			$\bar{1}_3\ \bar{2}_3\ \bar{2}_3\ \bar{0}_3\ \bar{1}_3\ \bar{1}_3\ \bar{2}_3$
	$a = 0, 4, 8$	$a = 1, 5, 6$	$a = 2, 3, 7$	

Table 3: Class patterns in Z_3

4 Structure of these sequences in Z_4

The class patterns in Z_4 (Table 4) are all rather complex in comparison with Z_3. Class $\bar{0}_4$ is dominant for the tetrahedral sequence which has mostly even numbers as elements. For the pentagonal sequence the $f(t)$ values for n were substituted into

$$K = f(n) = \tfrac{1}{2}(3n \pm 1) \tag{4.1}$$

with $n = a + 8t$. (Table 4) we get

$$K = \tfrac{1}{2}(a + 8t)((3a+1) + 24t). \tag{4.2}$$

Thus since an odd integer N^2 is given by

$$\begin{aligned} N^2 &= 4R_1 + 1 \\ &= 24K + 1 \end{aligned} \tag{4.3}$$

with $R_1 = 6K$. N may be found from K, using $n = f(t)$ (Tables 5, 6). For all classes we have the first order recurrence relation

$$N_{i+1} = N_i + 48 \tag{4.4}$$

Type	Class function for $n \in Z_4$				Class patterns
	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	
Triangular	$n = a + 8t$				$\bar{1}_4\ \bar{3}_4\ \bar{2}_4\ \bar{2}_4\ \bar{1}_4\ \bar{3}_4\ \bar{0}_4\ \bar{0}_4$
	$a = 0, 7$	$a = 1, 6$	$a = 3, 4$	$a = 2, 5$	Most members $3 \mid f(n)$. Predicts $3 \mid N^2$
Tetrahedral	$n = a + 8t$				$\bar{1}_4\ \bar{0}_4\ \bar{2}_4\ \bar{0}_4\ \bar{3}_4\ \bar{0}_4\ \bar{0}_4\ \bar{0}_4$
	$a = 2, 4, 6$	$a = 1$	$a = 3$	$a = 5$	Most members even
Pentagonal	$n = a + 8t$				$\bar{2}_4\ \bar{3}_4\ \bar{3}_4\ \bar{2}_4\ \bar{0}_4\ \bar{1}_4\ \bar{1}_4\ \bar{0}_4$
	$a = 0, 5$	$a = 6, 7$	$a = 1, 4$	$a = 2, 3$	Predicts rows of odd squares $3 \nmid N^2$
					$\bar{1}_4\ \bar{1}_4\ \bar{0}_4\ \bar{2}_4\ \bar{3}_4\ \bar{3}_4\ \bar{2}_4\ \bar{0}_4$
	$a = 8, 3$	$a = 1, 2$	$a = 4, 7$	$a = 5, 6$	Predicts rows of odd squares $3 \nmid N^2$
Pyramidal	$n = a + 8t$				$\bar{1}_4\ \bar{1}_4\ \bar{2}_4\ \bar{2}_4\ \bar{3}_4\ \bar{3}_4\ \bar{0}_4\ \bar{0}_4$
	$a = 7, 8$	$a = 1, 2$	$a = 3, 4$	$a = 5, 6$	

Table 4. Class patterns in Z_4

5 Final comments

The patterns of distribution in Z_3 and Z_4 of the various sequences have been given in terms of $n = f(t)$ and class sequences. The latter may be presented as vertical graphs or graphs may be constructed using the right end digits of each member sequentially with n. Such graphs have been presented for powers and primes [4, 5]. These new visual representations add another dimension [1] similar to spectral analysis used routinely in many of the sciences.

Whilst elements of the tetrahedral and pyramidal sequences commonly have 5 as a factor, division by 5 does not produce the link found when the triangular numbers are divided by 3 and become the pentagonal numbers alternating between $\frac{1}{2}n(3n-1)$ and $\frac{1}{2}n(3n+1)$.

t	$K\bar{0}_4$	N	$K\bar{1}_4$	N	$K\bar{2}_4$	N	$K\bar{3}_4$	N
	$(5+8t)(8+12t)$ $4t(1+24t)$		$(7+8t)(11+12t)$ $(3+4t)(19+214t)$		$(2+12t)(1+8t)$ $(2+4t)(13+24t)$		$(3+8t)(5+12t)$ $(1+4t)(7+24t)$	
0	40	31	77	43	2	7	15	19
	0	1	57	37	26	25	7	13
1	260	79	345	91	126	55	187	67
	100	49	301	85	222	73	155	61
2	672	127	805	139	442	103	551	115
	392	97	737	133	610	121	495	109
3	1276	175	1365	181	950	151	1107	163
	876	145	1457	187	1190	169	1027	157
4	2072	223	2185	229	1650	190	1855	211
	1552	193	2301	235	1962	217	1751	205

Table 5. $N \equiv N(K)$ where if $K = f(n) = \frac{1}{2}n(3n+1)$, then $N^2 = 24K + 1$

t	$K\bar{0}_4$	N	$K\bar{1}_4$	N	$K\bar{2}_4$	N	$K\bar{3}_4$	N
	$(3+8t)(4+12t)$ $(4+4t)(23+24t)$		$(1+12t)(1+8t)$ $(1+4t)(5+24t)$		$(2+4t)(11+24t)$ $(7+8t)(10+12t)$		$(5+8t)(7+12t)$ $(3+4t)(17+24t)$	
0	12	17	1	5	22	23	35	29
	92	47	5	11	70	41	51	35
1	176	65	117	53	210	71	247	77
	376	95	145	59	330	89	287	83
2	532	113	425	101	590	119	651	125
	852	143	477	107	782	137	715	131
3	1080	161	925	149	1162	167	1247	173
	1520	191	1001	155	1426	185	1335	179
4	1820	209	1617	197	1926	215	2035	221
	2380	239	1717	203	2262	233	2147	227

Table 6. $N \equiv N(K)$ where if $K = f(n) = \frac{1}{2}n(3n-1)$, then $N^2 = 24K + 1$

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