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Some properties and applications of a new arithmetic function in analytic number theory

Ramesh Kumar Muthumalai

Department of Mathematics, D.G. Vaishnav College
Chennai-600 106, Tamil Nadu, India
e-mail: $ramjan\_80@yahoo.com$

Abstract: We introduce an arithmetic function and study some of its properties analogous to Möbius function in analytic number theory. In addition, we derive some simple expressions to connect infinite series and infinite products through this new arithmetic function. Moreover, we study applications of this new function to derive simple identities on partition of an integer and some special infinite products in terms of multiplicative functions.
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1 Introduction

Analytic number theory is the branch of number theory which uses real and complex analysis to investigate various properties of arithmetic functions and prime numbers [1]. There is a variety of truly interesting arithmetic functions such as Euler's totient function, Möbius function, divisor function, etc.

Möbius function is a simple multiplicative function that arises in many different places in number theory. It plays an important role in the study of divisibility properties of integers. The classical Möbius function [1, 4, 5] is defined by

$$\begin{aligned} \mu(n) &= 1, && \text{If } n = 1 \\ &= (-1)^k, && \text{If } n \text{ is square free with } k \text{ distinct primes} \\ &= 0, && \text{Otherwise} \end{aligned} \tag{1.1}$$

Some of its fundamental properties are a remarkably simple formula for the divisor sum $\sum_{d|n} \mu(d)$, extended over the positive divisors of n, and Möbius inversion formula [1, 4, 5]. There are many generalizations of the Möbius function, as well as analogous functions described by different authors in number theory. The survey of many generalizations of Möbius function in Group theory, Lattice theory, Partially ordered sets, etc., are found in [7].

Infinite products play an important role in many branches of mathematics. In fact, they provide an elegant way of encoding and manipulating combinatorial identities. The product expansion of generating function of partition function is a familiar example [2, 3]. In this paper, we introduce an arithmetic function and study some of its properties analogous to Möbius function. Further more, we incorporate this new function applying it to infinite products, partition of an integer and expressions connecting with divisor functions in analytic number theory.

2 The new arithmetic function and its properties

Let us begin by defining the new arithmetic function on positive integers.

Definition 2.1. For any two positive integers n and p, define an arithmetic function $\upsilon_p(n)$ as follow as

$$\upsilon_p(n) = \begin{cases} 1, & \text{If } n = 1 \\ 2^{r-1}, & \text{If } n = p^r,\ r \in N,\ p > 1 \\ (-1)^k, & \text{If either } p \nmid n \text{ or } p = 1 \text{ and } n \text{ is} \\ & \text{square-free with } k \text{ distinct primes} \\ (-1)^k 2^{r-1}, & \text{If } n = p^r m,\ r \in N,\ p \nmid m \text{ and } m \text{ is} \\ & \text{square-free with } k \text{ distinct primes} \\ 0, & \text{Otherwise} \end{cases} \tag{2.1}$$

Remark 2.2. The following results are immediately derived from the definition of $\upsilon_p(n)$ and Möbius function,

1. If $p \nmid n$ and n is square free, then $\upsilon_p(n) = \mu(n)$.
2. If $n = p^r m, p \nmid m$ and m is square free, then $\upsilon_p(n) = 2^{r-1}\mu(m)$.
3. In particular $\upsilon_1(n) = \mu(n)$, $n \in N$.

Theorem 2.3. If p is any prime or $p = 1$, then υ_p is multiplicative.

Proof. It is clearly true when $p = 1$. Assume that p is any prime. Suppose that m and n are any two square-free integers with $(m, n) = 1$ and $p \nmid mn$, then by Remark 2.2 $\upsilon_p(mn) = \mu(mn)$. Since μ is multiplicative, υ_p is also multiplicative. In case $(p^r, n) = 1$ and n is square free with k distinct primes, then by (2.1) $\upsilon_p(p^r n) = 2^{r-1}(-1)^k = \upsilon_p(p^r)\upsilon_p(n)$. If $(p^r m, n) = 1$, $p \nmid mn$ and m, n are square free with k_1, k_2 distinct primes, then $\upsilon_p(p^r mn) = 2^{r-1}(-1)^{k_1+k_2} = \upsilon_p(p^r m)\upsilon_p(n)$. It is also true for other cases. Hence, υ_p is multiplicative. $\square$

Definition 2.4. For any two positive integers n and p, define

$$\delta_p(n) = \begin{cases} -1, & \text{If } p|n \text{ and } p > 1 \\ 1, & \text{If either } p \nmid n \text{ or } p = 1 \end{cases} \tag{2.2}$$

The following theorem gives an interesting property of $\upsilon_p(n)$ analogous to Möbius function.

Theorem 2.5. If p is any prime or $p = 1$ and $n \in N$, then

$$\textstyle\sum_{d|n} \delta_p(n/d)\upsilon_p(d) = \begin{cases} 1, & \textit{if } n = 1 \\ 0, & \textit{if } n > 1 \end{cases} \tag{2.3}$$

Proof. The formula is true if $n = 1$ or $p = 1$, assume that $n > 1$ and p is any prime. If $p \nmid n$ and n is square free, then $n = p_1 p_2 \ldots p_k$ ($p_1, p_2, \ldots, p_k$ are distinct primes different from p). Since $p \nmid n$, then $\delta_p(n/d) = 1$ and using Remark 2.2, we find that

$$\sum_{d|n} \delta_p(n/d)\upsilon_p(d) = \sum_{d|p_1p_2\ldots p_k} \upsilon_p(d) = \sum_{d|p_1p_2\ldots p_k} \mu(d) = 0$$

If $n = p^k$, then by Definition 2.4, we find that $\delta_p(p^i) = -1$, $\delta_p(1) = 1$ and using Definition 2.1, we obtain

$$\sum_{d|p^k} \delta_p(p^k/d)\upsilon_p(d) = -1 - \sum_{i=1}^{k-1} \upsilon_p(p^i) + \upsilon_p(p^k) = -1 - \sum_{i=1}^{k-1} 2^{i-1} + 2^{k-1} = 0$$

If $n = p^k m$, where m is square free integer and $p \nmid m$. Let d runs through divisors of $p^k m$. The summation can be splitted as follows

$$\sum_{d|p^k m} \delta_p(p^k m/d)\upsilon_p(d) = \sum_{w=0}^{k-1} \sum_{d|m} \delta_p(p^{k-w} m/d)\upsilon_p(p^w d) + \sum_{d|m} \delta_p(m/d)\upsilon_p(p^k d)$$

Since $p \nmid m$ and using (2.1) and (2.4), we find that

$$= -\sum_{d|m} \mu(d) - \sum_{w=1}^{k-1} 2^{w-1} \sum_{d|m} \mu(d) + 2^{k-1} \sum_{d|m} \mu(d) = 0$$

Further $\sum_{d|n} \delta_p(n/d)\upsilon_p(d) = 0$, for other values of n. □

Theorem 2.6. *For any prime p or $p = 1$, δ_p and υ_p are Dirichlet inverse to each other, i.e. $\delta_p^{-1} = \upsilon_p$ and $\upsilon_p^{-1} = \delta_p$.*

Proof. If I denotes identity function for all values of n, then using Theorem 2.5 we find that

$$I(n) = \sum_{d|n} \delta_p(n/d)\upsilon_p(d) \tag{2.4}$$

In the notation of Dirichlet multiplication [1, 5], this becomes

$$I = \delta_p * \upsilon_p \tag{2.5}$$

Hence, δ_p and υ_p are Dirichlet inverse to each other. □

The above property of δ_p and υ_p, along with the associative property of Dirichlet multiplication, enables us to give a simple proof of the next theorem.

Theorem 2.7. *Suppose that $f(n) = \sum_{d|n} \delta_p(n/d)g(d)$ if and only if $g(n) = \sum_{d|n} \upsilon_p(d)f(n/d)$ for prime p or $p = 1$.*

Proof. Assume that $f(n) = \sum_{d|n} \delta_p(n/d)g(d)$ is true. So that $f = g * \delta_p$. Multiplying by υ_p on both sides gives

$$f * \upsilon_p = (g * \delta_p) * \upsilon_p = g * (\delta_p * \upsilon_p) = g * I = g$$

Conversly, multiplication of $f * \upsilon_p = g$ by δ_p gives the proof. □

Theorem 2.8. *If f is completely multiplicative, then*

$$(\delta_p f)^{-1}(n) = \upsilon_p(n)f(n) \quad \textit{for all } n \geq 1 \tag{2.6}$$

Proof. Given that f is completely multiplicative and let $g(n) = \upsilon_p(n)f(n)$

$$
\begin{aligned}
(g * (\delta_p f))(n) &= \sum_{d|n} \upsilon_p(d)f(d)\delta_p\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right) \\
&= f(n)\sum_{d|n} \upsilon_p(d)\delta_p\left(\frac{n}{d}\right) \\
&= I(n)
\end{aligned}
$$

Since $f(1) = 1$ and $I(n) = 0$ for $n > 1$. Hence, $g = (\delta_p f)^{-1}$. $\square$

The following lemma connects the new arithemtic function $\upsilon_p(n)$ and infinite products.

Lemma 2.9. *For any positive integer $p > 1$, $x^2 \leq 1$ and $x^k \cos k\theta \neq 1, 2, 3, \ldots$*

$$
\sum_{k=1}^{\infty} \delta_p(k)x^k \frac{\cos k\theta}{k} = \frac{1}{2}\log\frac{\left(1 - 2x^p\cos p\theta + x^{2p}\right)^{2/p}}{1 - 2x\cos\theta + x^2} \tag{2.7}
$$

Proof. The LHS of (2.7) can be splitted as follow as

$$
\sum_{k=1}^{\infty} \delta_p(k)x^k \frac{\cos k\theta}{k} = \sum_{k=1, p\nmid k}^{\infty} x^k \frac{\cos k\theta}{k} - \sum_{k=1}^{\infty} x^{pk}\frac{\cos pk\theta}{pk}
$$

Now adding and subtracting $\sum_{k=1}^{\infty} x^{pk}\frac{\cos pk\theta}{pk}$ to the RHS of the above equation, and using the following identity [6, pp-52],

$$
\sum_{k=1}^{\infty} x^k \frac{\cos k\theta}{k} = -\frac{1}{2}\log\left(1 - 2x\cos\theta + x^2\right)
$$

and after simplification, we obtain (2.7). $\square$

Theorem 2.10. *If p is any prime, $x^2 \leq 1$ and $x^k \cos k\theta \neq 1, 2, 3, \ldots$, then*

$$
e^{2x\cos\theta} = \prod_{k=1}^{\infty}\left(\frac{\left(1 - 2x^{kp}\cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k\cos k\theta + x^{2k}}\right)^{\frac{\upsilon_p(k)}{k}} \tag{2.8}
$$

Proof. Let us consider the following infinite series

$$
\sum_{k=1}^{\infty}\frac{\upsilon_p(k)}{k}\log\frac{\left(1 - 2x^{kp}\cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k\cos k\theta + x^{2k}}
$$

where all υ_p are defined from (2.1). Now using the lemma 2.9, we find that

$$
= \sum_{k=1}^{\infty}\frac{\upsilon_p(k)}{k}\left(2\sum_{i=1}^{\infty}\delta_p(i)\frac{\cos ki\theta}{i}x^{ki}\right)
$$

Rearranging in ascending powers of x, using Theorem 2.5 and after simplification, we obtain (2.8). $\square$

Corollary 2.11. *If* $x^2 \leq 1$ *and* $x^k \cos k\theta \neq 1, 2, 3, \ldots,$ *then*

$$e^{2x\cos\theta} = \prod_{k=1}^{\infty} \left(1 - 2x^k \cos\theta + x^{2k}\right)^{-\frac{\upsilon_1(k)}{k}} \tag{2.9}$$

Proof. This can be easily found by using (2.7) and Theorem 2.5. □

Remark 2.12. For any prime p, the following infinite products are obtained as some special cases of Theorem 2.10 and Corollary 2.11

1. If $\theta = 0$ in (2.8), then

$$e^x = \prod_{k=1}^{\infty} \left(\frac{\left(1 - x^{kp}\right)^{2/p}}{1 - x^k} \right)^{\frac{\upsilon_p(k)}{k}}$$

2. If $\theta = 0$ in (2.9), then we obtain following infinite product which can be rewritten as Lambert series of Möbius function

$$e^x = \prod_{k=1}^{\infty} (1 - x^k)^{-\frac{\upsilon_1(k)}{k}}$$

3 Connecting infinite series and infinite products through $\nu_p(n)$

In this section, we derive simple expressions through $\nu_p(n)$ that connect infinite series and infinite products.

Theorem 3.1. *For* $x^2 \leq 1$, $x^k \cos k\theta \neq 1$, $k \in N$ *and* $P(x,\theta) = \sum_{i=1}^{\infty} c(i) x^i \cos i\theta$, *if* p *is any prime, then*

$$e^{2P(x,\theta)} = \prod_{k=1}^{\infty} \left(\frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos\theta + x^{2k}} \right)^{a(k)} \tag{3.1}$$

If $p = 1$, *then*

$$e^{-2P(x,\theta)} = \prod_{k=1}^{\infty} \left(1 - 2x^k \cos\theta + x^{2k}\right)^{a(k)} \tag{3.2}$$

where

$$a(k) = \sum_{d|k} c(d) \frac{\upsilon_p(k/d)}{k/d} \tag{3.3}$$

Proof. Given that $P(x,\theta) = \sum_{i=1}^{\infty} c(i) x^i \cos i\theta$, taking logarithm on both sides and using (2.9), we obtain

$$= \frac{1}{2} \sum_{i=1}^{\infty} c(i) \sum_{k=1}^{\infty} \upsilon_p(k) \log \frac{\left(1 - 2x^{ikp} \cos ikp\theta + x^{2kip}\right)^{2/p}}{1 - 2x^{ik} \cos ik\theta + x^{2ik}}$$

Rearranging the above, we have

$$= \frac{1}{2} \sum_{k=1}^{\infty} \left(\sum_{d|k} c(d) \frac{\upsilon_p(k/d)}{k/d} \right) \log \frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos k\theta + x^{2k}}$$

Assuming $a(k) = \sum_{d|k} c(d) \frac{\upsilon_p(k/d)}{k/d}$ and after simplification, we obtian (3.1). In a similar manner, we easily obtain (3.2) from (2.9). □

Remark 3.2. Putting $\theta = 0$ in (3.1) and (3.2), for any prime p, we obtain,

$$e^{P(x,0)} = \prod_{k=1}^{\infty} \left(\frac{\left(1 - x^{kp}\right)^{2/p}}{1 - x^k} \right)^{a(k)} \tag{3.4}$$

and if $p = 1$,

$$e^{-P(x,0)} = \prod_{k=1}^{\infty} \left(1 - x^k\right)^{a(k)} \tag{3.5}$$

Corollary 3.3. *If* $a(k) = \sum_{d|k} c(d) \frac{\upsilon_p(k/d)}{k/d}$, *then* $c(k) = \sum_{d|k} a(d) \frac{\delta_p(k/d)}{k/d}$

Proof. From Theorem 2.7, we find that $f(k) = \sum_{d|k} \upsilon_p(k/d) g(d)$. Comparing this with $a(k) = \sum_{d|k} c(d) \frac{\upsilon_p(k/d)}{k/d}$, we have $f(k) = ka(k)$ and $g(d) = dc(d)$, where $a(k)$ is function in k and $c(d)$ is function in d. Hence, $kc(k) = \sum_{d|k} a(d) d \delta_p(k/d)$. Now, simplifying this, we obtain the corollary. □

Example 3.4. We can express cosine function on $0 < x < 1$ as follow as

$$\cos x = (1 - x^2)^{-a(1)} (1 - x^4)^{-a(2)} (1 - x^6)^{-a(3)} \ldots \tag{3.6}$$

where $c(k) = \frac{2^{2k-1}(2^{2k}-1)}{k.2k!} |B_{2k}|$. It can be found by using (3.2) and the following infinite series [6]

$$\log\left(\frac{1}{\cos x}\right) = \frac{2(2^2-1)|B_2|}{1.2!} x^2 + \frac{2^3(2^4-1)|B_4|}{2.4!} x^4 + \frac{2^5(2^6-1)|B_6|}{3.6!} x^6 + \ldots$$

Theorem 3.5. *For any prime* p *or* $p = 1$, $x \neq 0$, $x^2 \leq 1$ *and* $0 < \theta < 2\pi$, *if* $T(\theta) = \sum_{k=1}^{\infty} t(k) \cos k\theta$ *then*

$$e^{2T(\theta)} = \prod_{k=1}^{\infty} \left(\frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos k\theta + x^{2k}} \right)^{t'(k)} \tag{3.7}$$

where $t'(k) = \sum_{d|k} \frac{t(d)}{x^d} \frac{\upsilon_p(k/d)}{k/d}$

Proof. Given that $T(\theta) = \sum_{i=1}^{\infty} t(i) \cos i\theta$, using (2.7), we get

$$T(\theta) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{t(i)}{x^i} \sum_{k=1}^{\infty} \upsilon_p(k) \log \frac{\left(1 - 2x^{ikp} \cos ikp\theta + x^{2ikp}\right)^{2/p}}{1 - 2x^{ik} \cos ik\theta + x^{2ik}}$$

After simplification, we obtain

$$= \frac{1}{2} \sum_{k=1}^{\infty} \log \left(\frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos k\theta + x^{2k}} \right) \sum_{d|k} \frac{t(d)}{x^d} \frac{\upsilon_p(k/d)}{k/d}$$

Setting $t'(k) = \sum_{d|k} \frac{t(d)}{x^d} \frac{\upsilon_p(k/d)}{k/d}$ and transforming from logarithmic form to exponential form, we obtain (3.7). □

Example 3.6. We can write the infinite series $\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\log 2 \sin \frac{x}{2}$ on $[0 < \theta < 2\pi]$ found in [6], $x \neq 0$ as infinite produts as follow as

$$\csc^2 \frac{\theta}{2} = 4 \prod_{k=1}^{\infty} \left(\frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos \theta + x^{2k}} \right)^{t'(i)} \tag{3.8}$$

where $t'(k) = \frac{1}{k} \sum_{d|k} \frac{\upsilon_p(k/d)}{x^d}$

4 Applications of the new arithmetic function ν_p

Let us start from the following lemmas that connect infinite series containing divisor sum that extends over positive integers with infinite products.

Lemma 4.1. *For any prime p, $x^2 \leq 1$ and $x^k \cos k\theta \neq 1$, $k \in N$, if $\varrho_v^{(p)}(k) = \sum_{d^v|k} d^v \delta_p(k/d^v)$ and $\alpha_p(x,\theta) = \sum_{k=1}^{\infty} \varrho_v^{(p)}(k) \frac{x^k}{k} \cos k\theta$, then*

$$e^{2\alpha_p(x,\theta)} = \prod_{k=1}^{\infty} \frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos\theta + x^{2k}} \tag{4.1}$$

Proof. Setting $a(k) = 1$, for $k = 1^v, 2^v, 3^v, \ldots$ and $a(k) = 0$ for other k's then by Corollary 3.3, we find that

$$c(k) = \sum_{d^v|k} \frac{\delta_p(k/d^v)}{k/d^v} = \frac{1}{k} \sum_{d^v|k} d^v \delta_p(k/d^v) = \frac{1}{k} \varrho_v^{(p)}(k) \quad \text{(say)}$$

Using Theorem (3.1) and setting $P(x,\theta) = \alpha_p(x,\theta)$, we obtain (4.1). □

Lemma 4.2. *For $x^2 \leq 1$, $x^k \cos k\theta \neq 1$, $k \in N$, if*
$\rho_v(k) = \sum_{d^v|k} d^v$
and
$\beta(x,\theta) = \sum_{k=1}^{\infty} \rho_v(k) \frac{x^k}{k} \cos k\theta$,
then

$$e^{-2\beta(x,\theta)} = \prod_{k=1}^{\infty} 1 - 2x^k \cos k\theta + x^{2k} \tag{4.2}$$

Proof. Setting $a(k) = 1$, for $k = 1^v, 2^v, 3^v, \ldots$ and $a(k) = 0$ for other k's then by Corollary 3.3, we find that $\rho_v(k) = \sum_{d^v|k} d^v$. Using Theorem 3.2, we obtain (4.2). □

4.1 Partition of an interger

Let us assume $\alpha_p(x) = \sum_{k=1}^{\infty} \varrho_v^{(p)}(k) \frac{x^k}{k}$ and $\beta(x) = \sum_{k=1}^{\infty} \rho_v(k) \frac{x^k}{k}$ which are special cases of $\alpha_p(x,\theta)$ and $\beta(x,\theta)$ at $\theta = 0$.

Theorem 4.3. *Let $p^{(v)}(n)$ denote number of partition of n into perfect v-th power, for $|x| < 1$ and $v > 0$*

$$1 + \sum_{n=0}^{\infty} p^{(v)}(n) x^n = e^{\beta(x)} \tag{4.3}$$

and

$$p^{(v)}(n) = \frac{1}{n!} \left. \frac{d^n \left(e^{\beta(x)}\right)}{dx^n} \right|_{x=0} \tag{4.4}$$

Proof. We know that

$$1 + \sum_{n=1}^{\infty} p^{(v)}(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^{k^v}} \tag{4.5}$$

Putting $\theta = 0$ and setting $\beta(x,0) = \beta(x)$ in Lemma 4.2, we find that

$$e^{\beta(x)} = \prod_{k=1}^{\infty} \frac{1}{1 - x^{k^v}}$$

Equating above equations, we get (4.3). Now, differentiating n times with respect to x and setting $x = 0$, we obtain (4.4). $\square$

Theorem 4.3 clarifies that $e^{\beta(x)}$ is a generating function for partition of an integer into perfect v-th power in terms of divisor functions.

Corollary 4.4. *For $v, k \in N$*

$$np^{(v)}(n) = \sum_{k=1}^{n} \rho_v(k) p^{(v)}(n-k) \tag{4.6}$$

Proof. Taking $\log$ on both sides of (4.3) and differentiating with respect to x,

$$\sum_{n=1}^{\infty} p_v(n) n x^{n-1} = \left(\sum_{n=1}^{\infty} \rho_v(n) n x^{n-1} \right) \left(1 + \sum_{n=1}^{\infty} p_v(n) x^n \right)$$

Simplifying the above expression and equating coefficients of x, we obtain (4.6). $\square$

Theorem 4.5. *For $v, k \in N$*

$$p^{(v)}(n) = \sum \frac{1}{i!j!h!\ldots k!} \frac{\rho_v(1)^i \rho_v(2)^j \rho_v(3)^h \ldots \rho_v(l)^k}{1^i 2^j 3^h \ldots l^k} \tag{4.7}$$

where $\rho_v(k) = \sum_{d^v|k} d^v$ and the symbol $\sum$ indicates summation over all solutions in non negative integers of the equation $i + 2j + 3h + \ldots + lk = n$.

Proof. This is immediatly from (4.3), by expanding through the exponential function, simplifying and equating respective coefficients of x^n. $\square$

Remark 4.6. The following identity is obtained by putting $v = 1$ in (4.7)

$$p(n) = \sum \frac{1}{i!j!h!\ldots k!} \frac{\sigma_1^i(1)\sigma_1^j(2)\sigma_1^h(3)\ldots\sigma_1^k(l)}{1^i 2^j 3^h \ldots l^k}$$

The symbol $\sum$ indicates summation over all solutions in non negative integers of the equation $i + 2j + 3h + \ldots + lk = n$.

Theorem 4.7. *If $|r| < 1$, $\beta_c(r, \alpha) = \sum_{k=1}^{\infty} \rho_v(k) \cos k\alpha$ and $\beta_s(r, \alpha) = \sum_{k=1}^{\infty} \rho_v(k) \sin k\alpha$, then*

$$p^{(v)}(m) = \frac{1}{\pi r^m} \int_0^{2\pi} e^{\beta_c(r,\alpha)} \beta_s(r, \alpha) \cos m\alpha d\alpha \tag{4.8}$$

$$= \frac{1}{\pi r^m} \int_0^{2\pi} e^{\beta_c(r,\alpha)} \beta_s(r, \alpha) \sin m\alpha d\alpha \tag{4.9}$$

Proof. If we take $x = re^{i\alpha}$ and $|r| < 1$ then

$$\beta(re^{i\alpha}) = \sum_{k=1}^{\infty} \rho_v(k) r^k \cos k\alpha + i \sum_{k=1}^{\infty} \rho_v(k) r^k \sin k\alpha$$

Setting $\beta_s(r, \alpha) = \sum_{k=1}^{\infty} \rho_v(k) r^k \sin k\alpha$ and $\beta_c(r, \alpha) = \sum_{k=1}^{\infty} \rho_v(k) r^k \cos k\alpha$, then we have $\beta(re^{i\alpha}) = \beta_c(r, \alpha) + i\beta_s(r, \alpha)$. Using this in (4.5) and comparing real and imaginary parts, we obtain

$$1 + \sum_{n=1}^{\infty} p^{(v)}(n) r^n \cos n\alpha = e^{\beta_c(r,\alpha)} \cos \beta_s(r, \alpha) \tag{4.10}$$

$$\sum_{n=1}^{\infty} p^{(v)}(n) r^n \sin n\alpha = e^{\beta_c(r,\alpha)} \sin \beta_s(r,\alpha) \tag{4.11}$$

Multiplying by $\cos m\alpha$ and $\sin m\alpha$, $m = 1, 2, 3, \ldots$, integrating on the interval $(0, 2\pi)$. We obtain (4.8) and (4.9). $\square$

Remark 4.8. Now, squaring and adding (4.10) and (4.11), we have

$$\left(1 + \sum_{n=1}^{\infty} p^{(v)}(n) r^n \cos n\alpha\right)^2 + \left(\sum_{n=1}^{\infty} p^{(v)}(n) r^n \sin n\alpha\right)^2 = e^{2\beta_c(r,\alpha)} \tag{4.12}$$

Simplifying (4.12) and integrating on $(0, 2\pi)$, we have

$$1 + \frac{1}{2} \sum_{n=1}^{\infty} p^{(v)}(n)^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} e^{2\beta_c(r,\alpha)} d\alpha \tag{4.13}$$

We can find similar generating functions and identities for various partition functions. For example, if $\beta(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \rho(k)$ and $\rho(k) = \sum_{d|k, odd\ d} d$, then $e^{\beta(x)}$ is the generating function for the number of partitions of n (say $p(n)$) into parts which are odd. Similarly, $\rho(k) = \sum_{d|k, even\ d} d$, then $e^{\beta(x)}$ generates the number of partitions of n into parts which are even and so on. Hence, we find generalized identities for the partition function, as follows

$$np(n) = \sum_{k=1}^{n} \rho(k) p(n-k)$$

and

$$p(n) = \sum \frac{1}{i!j!h!\ldots k!} \frac{\rho^i(1)\rho^j(2)\rho^h(3)\ldots\rho^k(l)}{1^i 2^j 3^h \ldots l^k}$$

The symbol $\sum$ indicates summation over all solutions in non negative integers of the equation $i + 2j + 3h + \ldots + lk = n$.

To find similar generating functions and identities for number of partitions of n into parts which are unequal, set $p = 2$, $\theta = 0$ and replace $\alpha_2(x, 0)$ by $\alpha_2(x)$. Hence, we have $e^{\alpha_2(x)} = \prod_{k=1}^{\infty} 1 + x^k$ (where $\alpha_2(x) = \sum_{k=0}^{\infty} \varrho_v^{(2)}(k) \frac{x^k}{k}$ and $\varrho_v^{(2)}(k) = \sum_{d|k} d\delta_2(k/d)$). Now, $e^{\alpha_2(x)}$ generates the number of partitions of n into parts which are unequal. Similarly, we can find generating functions of other cases such as distinct primes, odd, etc. So, we generalize the following identities

$$np(n) = \sum_{k=1}^{n} \varrho_v^{(2)}(k) p(n-k)$$

and

$$p(n) = \sum \frac{1}{i!j!h!\ldots k!} \frac{\varrho_v^{(2)}(1)^i \varrho_v^{(2)}(2)^j \varrho_v^{(2)}(3)^h \ldots \varrho_v^{(2)}(l)^k}{1^i 2^j 3^h \ldots l^k}$$

4.2 Some special infinite products

If p is any positive integer, then varying the values of $a(k)$ by Corollary 3.3, we find some interesting infinite products connecting with divisor sum extended over positive integers. Let us consider for any prime p,

$$e^{2\alpha_p(x,\theta)} = \prod_{k=1}^{\infty} \left(\frac{\left(1 - 2x^{kp} \cos kp\theta + x^{2kp}\right)^{2/p}}{1 - 2x^k \cos \theta + x^{2k}} \right)^{a(k)} \tag{4.14}$$

and when $p = 1$

$$e^{-2\beta(x,\theta)} = \prod_{k=1}^{\infty} \left(1 - 2x^k \cos k\theta + x^{2p}\right)^{a(k)} \tag{4.15}$$

The values of $\alpha(x,\theta)$ and $\beta(x,\theta)$ for various a's are listed below:

1. If $a(k) = \frac{1}{k^s}$, $k \in N$

$$\alpha_p(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{k} \sum_{d|k} \frac{\delta_p(k/d)}{d^{s-1}}$$

   and

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{\sigma_{s-1}(x)}{k} x^k \cos k\theta.$$

2. Let $a(k) = \frac{1}{k^s}$, if k is prime and $a(k) = 0$ for other k's

$$\alpha_p(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{k} \sum_{\substack{d|k \\ prime\ p}} \frac{\delta_p(k/d)}{d^{s-1}}$$

   and

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{k} \sum_{\substack{d|k \\ prime\ p}} \frac{1}{d^{s-1}}.$$

3. Let $a(k) = \phi(k)/k^s$, where ϕ is Euler's totient function

$$\alpha_p(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{k} \sum_{d|k} \phi(d) \frac{\delta_p(k/d)}{d^{s-1}}$$

   and

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{k} \sum_{d|k} \frac{\phi(d)}{d^{s-1}}.$$

   In particular when $s = 1$, and using the identity $\sum_{d|n} \phi(d) = n$, we have $\beta(x,\theta) = \sum_{k=1}^{\infty} x^k \cos k\theta$

4. Let $a(k) = \Lambda(k)/k$ (where Λ is Mangoldt's function) and using the identity $\sum_{d|n} \Lambda(d) = \log n$, we have

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{\log k}{k} x^k \cos k\theta.$$

5. Let $a(k) = \mu(k)/k^2$ and using the identity $\phi(n)/n = \sum_{d|n} \frac{\mu(d)}{d}$, we have

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k^2} x^k \cos k\theta.$$

6. Let $a(k) = \frac{\mu^2(k)}{k\phi(k)}$ and using the identity $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}$, we have

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{\phi(k)}.$$

7. Let $a(k) = \frac{\mu^2(k)}{k}$ and using the identity $2^{v(n)} = \sum_{d|n} \mu^2(d)$ where $v(1) = 0$ and $v(n) = k$ if $n = p_1^{a_1} \dots p_k^{a_k}$, we have

$$\beta(x,\theta) = \sum_{k=1}^{\infty} \frac{2^{v(k)}}{k} x^k \cos k\theta.$$

5 Conclusion

In conclusion, we note that some properties and applications of a new arithmetic function $\upsilon_p(n)$ have been developed in this article. Firstly, we have introduced $\upsilon_p(n)$ and proved some of its properties analogous to Möbius function. Secondly, we have derived some simpler infinite products of exponential function and some new expressions to connect infinite series and infinite products in terms of $\upsilon_p(n)$. Finally, we have developed some applications to generate functions of various partition functions of an integer, simple partition identities and special infinite products in terms of multiplicative functions.

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