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Uniqueness of the extension of the $D(4k^2)$-triple $\{k^2-4, k^2, 4k^2-4\}$

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Abstract: Let n be a nonzero integer. A set of m distinct positive integers is called a $D(n)$-m-tuple if the product of any two of them increased by n is a perfect square. Let k be an integer greater than two. In this paper, we show that if $\{k^2-4, k^2, 4k^2-4, d\}$ is a $D(4k^2)$-quadruple, then $d = 4k^4 - 8k^2$.

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1 Introduction

Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \dots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or a $D(n)$-m-tuple, if $a_ia_j + n$ is a perfect square for each i, j with $1 \le i < j \le m$.

The first example of a $D(1)$-quadruple was found by Fermat, which was the set $\{1, 3, 8, 120\}$. Baker and Davenport ([1]) showed that $\{1, 3, 8, 120\}$ cannot be extended to a $D(1)$-quintuple. There are several generalizations of this result (see [17] and its references); for example, $\{k-1, k+1\}$ cannot be extended to a $D(1)$-quintuple ([14]). A folklore conjecture says that there does not exist a $D(1)$-quintuple, and Dujella ([7]) showed that there does not exist a $D(1)$-sextuple and that there exist only finitely many $D(1)$-quintuples.

The $n = 4$ case can be considered in the same way as the $n = 1$ case (see [12] and its references) and it is conjectured that there does not exist a $D(4)$-quintuple. The following is a stronger version of this conjecture.

Conjecture 1.1. ([10, Conjecture 1]) *If $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$, then $d = a + b + c + (abc + rst)/2$, where r, s, t are positive integers defined by $ab + 4 = r^2,\ ac + 4 = s^2,\ bc + 4 = t^2$.*

In general, if $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple ([3, 16, 19]). Dujella ([4]) showed that if $n \not\equiv 2 \pmod 4$ and if $n \notin S := \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple, and conjectured the following.

Conjecture 1.2. ([5]) *There does not exist a $D(n)$-quadruple for $n \in S$.*

There are several results supporting Conjecture 1.2 for the $n = -1$ case (see [8] and its references), and the validity of Conjecture 1.2 for $n = -1$ implies the one for $n = -4$ ([4, Remark 3]). For $n \notin \{\pm 1, \pm 4\}$, it is not easy to show either nonexistence or uniqueness of extension of a $D(n)$-triple, unless an argument using congruences modulo a power of 2 works. This is why we have to know the fundamental solutions of at least two of the Pell equations

$$ay^2 - bx^2 = 1,\ az^2 - cx^2 = 1,\ bz^2 - cy^2 = 1.$$

The first author ([13]) showed that the $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ with $|k|$ prime cannot be extended to a $D(4k)$-quadruple. Moreover, he ([15]) proved that the $D(\mp k^2)$-triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$-quintuple. In either case, ab and ac are of Richaud-Degert type ([20]), which gives the fundamental solutions of the corresponding Pell equations. Furthermore, the Padé approximation method (a theorem of Bennett ([2]) or of Rickert ([21])) can work for the $D(\mp k^2)$-triple. Since $D(\pm 4)$-tuples have similar properties to $D(\pm 1)$-tuples as mentioned above, it is natural to ask whether the same is valid for $D(\pm 4k^2)$-tuples and $D(\pm k^2)$-tuples. This leads us to consider the $D(\mp 4k^2)$-triple $\{k^2, k^2 \pm 4, 4k^2 \pm 4\}$.

Suppose that $\{k^2, k^2 \pm 4, 4k^2 \pm 4, d\}$ is a $D(\mp 4k^2)$-quadruple. If k is even, say $k = 2k'$, this is equivalent to that $\{(k')^2, (k')^2 \pm 1, 4(k')^2 \pm 1, d'\}$ is a $D(\mp (k')^2)$-quadruple with some integer d'. It follows from Theorems 1.4 and 1.5 in [15] that we may assume that k is odd. Then, since $-4k^2 \equiv 12 \pmod{16}$, the set $\{k^2, k^2+4, 4k^2+4, d\}$ cannot be a $D(-4k^2)$-quadruple by Remark 3 in [4], stating that if $\{a_1, a_2, a_3, a_4\}$ is a $D(16l+12)$-quadruple with some integer l, then every a_i is even. Therefore, the $D(4k^2)$-quadruple $\{k^2-4, k^2, 4k^2-4, d\}$ with k odd is only to be considered. Our theorem in this paper is the following.

Theorem 1.3. *Let k be an integer greater than two. If $\{k^2-4, k^2, 4k^2-4, d\}$ is a $D(4k^2)$-quadruple, then $d = 4k^4 - 8k^2$.*

Since Theorem 1.3 with k even follows immediately from Theorem 1.4 in [15], we will assume that k is odd throughout this paper.

The quadruple in Theorem 1.3 can be interpreted as the quadruple $\{k^2-4, k^2, x_{1,0}, x_{2,0}\}$ in [4], where $x_{n,m}$ and $y_{n,m}$ are double sequences satisfying $x_{n,0} = (y_{n,0}^2 - 4k^2)/(k^2-4)$ and $y_{0,0} = 2k,\ y_{1,0} = 2k^2-4,\ y_{n+1,0} = ky_{n,0} - y_{n-1,0}$. Another interpretation of the quadruple is to regard it as an analogue of the quadruple in Conjecture 1.1. More generally, if $\{a, k^2b, c\}$ is a $D(4k^2)$-triple, then $\{a, k^2b, c, d\}$ is a $D(4k^2)$-quadruple with $d = a + k^2b + c + (abc + rst)/2$, where r, s, t are positive integers defined by $ab + 4 = r^2,\ ac + 4k^2 = s^2,\ bc + 4 = t^2$. Thus, Theorem 1.3 gives an example for which an analogue of Conjecture 1.1 holds. Note that this analogy does not hold in general. Filipin ([11, Theorem 3.10]) showed that if $\{1, 20, 33, d\}$ is a $D(16)$-quadruple, then $d = 105$ or 273.

The organization of this paper is as follows. In Section 1, we transform the problem into a system of Diophantine equations, whose solution can be expressed as the intersection of two recurrence sequences. The congruence method due to Dujella then gives a lower bound for the number of terms. In Section 3, the lower bound and the theorem of Bennett together yield $k \leq 511$. Finally, in Section 4 using the reduction method ([1, 9]) we arrive at a contradiction for each k with $k \leq 511$.

2 A lower bound for solutions

Let k be an odd integer greater than two. Suppose that $\{k^2-4, k^2, 4k^2-4, d\}$ is a $D(4k^2)$-quadruple. Then, there exist positive integers x, y', z' such that

$$(k^2-4)d + 4k^2 = x^2,\ k^2 d + 4k^2 = (y')^2,\ (4k^2-4)d + 4k^2 = (z')^2. \tag{2.1}$$

Clearly we have $y' \equiv 0 \pmod{k}$ and $z' \equiv 0 \pmod{2}$, which enable us to write $y' = ky$ and $z' = 2z$ with positive integers y and z. Eliminating d from (2.1), we obtain the system of Diophantine equations

$$x^2 - (k^2-4)y^2 = 16, \tag{2.2}$$
$$z^2 - (k^2-1)y^2 = 4 - 3k^2. \tag{2.3}$$

Since k is odd, $k^2 - 4 \equiv 5 \pmod{8}$. Hence, (2.2) implies that both x and y are even, say $x = 2X$ and $y = 2Y$. Then, (2.2) can be rewritten as $X^2 - (k^2-4)Y^2 = 4$. The positive solutions of this Pell equation have the form

$$\frac{X + Y\sqrt{k^2-4}}{2} = \left(\frac{k+\sqrt{k^2-4}}{2}\right)^m$$

and hence, the positive solutions of (2.2) have the form

$$x + y\sqrt{k^2-4} = 4\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m \qquad \text{with nonnegative integers } m. \tag{2.4}$$

The positive solutions of (2.3) can be expressed as follows.

Lemma 2.1. *Let (z, y) be a positive solution of the Diophantine equation (2.3). Then, there exist a nonnegative integer n and a solution (z_0, y_0) of (2.3) such that*

$$z + y\sqrt{k^2-1} = (z_0 + y_0\sqrt{k^2-1})(k + \sqrt{k^2-1})^n \tag{2.5}$$

with

$$|z_0| < \sqrt{\frac{3}{2}k^3}, \quad 0 < y_0 < \sqrt{2k}. \tag{2.6}$$

Proof. We omit the proof, since it proceeds along the same lines as the proof of Lemma 1 in [6] or Lemma 3.1 in [15]. □

Lemma 2.2. *If $v_m = w_n$ has a solution, then m is odd, n is even and $z_0 = \pm k$, $y_0 = 2$.*

Proof. By (2.4) and (2.5), we may write $y = v_m = w_n$, where

$$v_0 = 0,\ v_1 = 2,\ v_{m+2} = kv_{m+1} - v_m \tag{2.7}$$

and

$$w_0 = y_0,\ w_1 = ky_0 + z_0,\ w_{n+2} = 2kw_{n+1} - w_n. \tag{2.8}$$

Hence,

$$(v_m \pmod{2k})_{m\geq 0} = (0, 2, 0, -2, 0, 2, \dots)$$

and

$$(w_n \pmod{2k})_{n\geq 0} = (y_0, ky_0 + z_0, -y_0, -ky_0 - z_0, y_0, \dots).$$

Suppose that $v_m \equiv 0 \pmod{2k}$. Then either $y_0 \equiv 0 \pmod{2k}$ or $ky_0 + z_0 \equiv 0 \pmod{2k}$. By (2.6), we must have $ky_0 + z_0 \equiv 0 \pmod{2k}$. Then, $z_0 \equiv 0 \pmod{k}$ implies $y_0^2 \equiv 4 \pmod{k^2}$. It follows from (2.6) that $y_0^2 = 4$, which implies that z_0 is even. This contradicts (2.3). Hence, m is odd and $v_m \equiv \pm 2 \pmod{2k}$. Then, either $y_0 \equiv \pm 2 \pmod{2k}$ or $ky_0 + z_0 \equiv \pm 2 \pmod{2k}$. If $ky_0 + z_0 \equiv \pm 2 \pmod{2k}$, then $z_0 \equiv \pm 2 \pmod{k}$, say $z_0 = \alpha k \pm 2$, where α is an integer with $|\alpha| < \sqrt{3k/2} + 2/k$ by (2.6). Thus, $y_0^2 \equiv \pm 4\alpha k \pmod{k^2}$ and by (2.6) we have $k^2 - 4|\alpha|k$. If $k \geq 29$, then $y_0^2 = k^2 - 4|\alpha|k > 2k$, which contradicts (2.6). The only odd integers k with $3 \leq k \leq 27$ such that (z_0, y_0) with y_0 odd is a solution of (2.3) satisfying (2.6) are 7 and 19, and then $(z_0, y_0) = (17, 3)$ and $(89, 5)$, respectively (note that y_0 must be odd, since y_0 and z_0 now have the same parity and z_0 is odd). However, in either case $z_0 \pm 2 \not\equiv 0 \pmod{k}$, that is, z_0 is not of the form $\alpha k \pm 2$. Therefore, n is even and $y_0 \equiv \pm 2 \pmod{2k}$. It follows from (2.6) that $y_0 = 2$ and $z_0 = \pm k$. □

Now one can easily obtain the following three lemmas in the same ways as Lemmas 4.2 to 4.4 in [15].

Lemma 2.3. *If $v_m = w_n$ has a solution with $n \geq 2$ in the case of $z_0 = k$ or with $n \geq 6$ in the case of $z_0 = -k$, then $n \geq 2k^2 - 6$.*

Lemma 2.4. *If $y = w_n$, then $\log y > (n-1)\log(1.942k)$.*

Lemma 2.5. *If $v_m = w_n$ has a solution with $n \neq 0$, then $m > n$. Moreover, if $k \geq 5$, then $m \leq 2n$; if $k = 3$, then $m \leq 3n$.*

3 Application of a theorem of Bennett

Let $\theta_1 = \sqrt{1 - 4/k^2}$ and $\theta_2 = \sqrt{1 - 1/k^2}$. Then, a theorem ([2, Theorem 3.2]) of Bennett can be applied if $k^2 > \max\{4^9, 1^9\}$, that is, $k > 512$.

Theorem 3.1. *If $k > 512$, then the numbers θ_1 and θ_2 satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > (2680k^2)^{-1}q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(660k^2)}{\log(0.0116k^4)} < 2.$$

The following lemma is similar to Lemma 3.6 in [15].

Lemma 3.2. *All positive solutions of the system of Diophantine equations* (2.2) *and* (2.3) *satisfy*

$$\max\left\{\left|\theta_1 - \frac{x}{ky}\right|, \left|\theta_2 - \frac{z}{ky}\right|\right\} < \frac{2.2}{y^2}.$$

Combining Theorem 3.1 and Lemma 3.2, one can bound $\log y$ above in terms of k.

Lemma 3.3. *Suppose that* $\{k^2-4, k^4, 4k^2-4, d\}$ *is a* $D(4k^2)$*-quadruple with* $k > 512$*. Then,*

$$\log y < \frac{8\log(8.77k)\log(0.329k)}{\log(0.00419k)}.$$

Proof. Applying Theorem 3.1 with $p_1 = x$, $p_2 = z$, $q = ky$, we see from Lemma 3.2 that

$$(2680k^2)^{-1}(ky)^{-\lambda} < \frac{2.2}{y^2}.$$

Noting $\lambda < 2$, one can easily see that the desired inequality holds. □

Proposition 3.4. *Suppose that* $\{k^2-4, k^2, 4k^2-4, d\}$ *is a* $D(4k^2)$*-quadruple with* $d \neq 4k^4 - 8k^2$*. Then,* $k \leq 511$*.*

Proof. If $n = 0$, then $d = 0$, which cannot be an element in a $D(4k^2)$-quadruple. If $n = 2$ with $z_0 = -k$, then $d = 4k^4 - 8k^2$. Denoting w_4 with $z_0 = -k$ by w_4^- , we easily see that

$$v_5 < w_4^- < v_7;$$

indeed, $w_4^- = 8k^4 - 12k^2 + 2,\ v_5 = 2k^4 - 6k^2 + 2,\ v_7 = 2k^6 - 10k^4 + 12k^2 - 2$. Since m is odd and n is even by Lemma 2.2, we may apply Lemma 2.3. Suppose that $k \geq 513$. Then, Lemmas 2.3, 2.4 and 3.3 together imply that

$$\frac{2k^2-7}{8} < \frac{\log(8.77k)\log(0.329k)}{\log(1.942k)\log(0.00419k)} =: f(k). \tag{3.1}$$

Since $f(k)$ is a decreasing function for $k \geq 513$, we have $f(k) \leq f(513) < 9$, which contradicts (3.1). □

4 The reduction method

By the standard method (see, e.g., [15, Lemma 3.9]), the following estimate on the linear form in three logarithms is obtained.

Lemma 4.1. *If* $v_m = w_n$ *has a solution with* $n \neq 0$*, then*

$$0 < m\log\alpha_1 - n\log\alpha_2 + \log\alpha_3 < 10\alpha_2^{-m'}, \tag{4.1}$$

where

$$\alpha_1 = \frac{k+\sqrt{k^2-4}}{2},\ \alpha_2 = k + \sqrt{k^2-1},\ \alpha_3 = \frac{4\sqrt{k^2-1}}{\sqrt{k^2-4}(2\sqrt{k^2-1} \pm k)},$$

and $m' = m$ *if* $k \geq 5$; $m' = 2m/3$ *if* $k = 3$.

The following theorem of Matveev gives an upper bound for m.

Theorem 4.2. ([18]) *Let Λ be a linear form in logarithms of l multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ ($b_l \neq 0$). Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \leq j \leq l$. Define the numbers D, A_j ($1 \leq j \leq l$) and B by $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_l) : \mathbb{Q}]$, $A_j = \max\{Dh(\alpha_j), |\log \alpha_j|\}$, $B = \max\{1, \max\{|b_j|A_j/A_l;\ 1 \leq j \leq l\}\}$. Then,*

$$\log |\Lambda| > -C(l)C_0W_0D^2\Omega,$$

where

$$\begin{aligned}
C(l) &= \frac{8}{(l-1)!}(l+2)(2l+3)(4\,\mathrm{e}(l+1))^{l+1}, \\
C_0 &= \log(\mathrm{e}^{4.4l+7}\, l^{5.5} D^2 \log(\mathrm{e}\, D)), \\
W_0 &= \log(1.5\,\mathrm{e}\, BD \log(\mathrm{e}\, D)), \quad \Omega = A_1 \cdots A_l.
\end{aligned}$$

Proposition 4.3. *Suppose that $\{k^2-4, k^2, 4k^2-4, d\}$ is a $D(4k^2)$-quadruple. Then $m < 10^{17}$.*

Proof. We apply Theorem 4.2 with $l = 3$, $D = 4$, $b_1 = m$, $b_2 = -n$, $b_3 = 1$. We have $A_1 = 2\log \alpha_1 < 2 \log k$ and $A_2 = 2\log\alpha_2 < 2\log(2k)$. Since the minimal polynomial of α_3 over $\mathbb{Z}$ is

$$(k^2-4)^2(3k^2-4)^2X^4 - 32(k^2-1)(k^2-4)(5k^2-4)X^2 + 256(k^2-1)^2$$

up to a multiple of a constant, and $\gcd((k^2-4)(3k^2-4), 16(k^2-1))$ divides 3, the leading coefficient a_0 of the minimal polynomial of α_3 over $\mathbb{Z}$ satisfies

$$\frac{(k^2-4)^2(3k^2-4)^2}{9} \leq a_0 \leq (k^2-4)^2(3k^2-4)^2.$$

Since α_3 with the minus sign is less than 1.91 and α_3 with the plus sign is less than 1, we have

$$\begin{aligned}
A_3 = 4h(\alpha_3) &< \log(1.91^2(k^2-4)^2(3k^2-4)^2) < 8\log(1.55k), \\
A_3 &> \log\left(\frac{(k^2-4)^2(3k^2-4)^2}{9}\right) > 8\log(0.829k).
\end{aligned}$$

Hence, we obtain the following:

$$\begin{aligned}
B &< \max\left\{\frac{2m\log k}{8\log(0.829k)}, \frac{2n\log(2k)}{8\log(0.829k)}\right\} < \frac{m\log(2k)}{4\log(0.829k)} < 0.5m, \\
C(3) &= \frac{8}{2!}\cdot 5 \cdot 9(16\,\mathrm{e})^4 < 6.45 \cdot 10^8, \\
C_0 &= \log\left(\mathrm{e}^{4.4\cdot 3+7} \cdot 3^{5.5} \cdot 16\log(4\,\mathrm{e})\right) < 29.9, \\
W_0 &= \log(1.5\,\mathrm{e}\, B \cdot 4\log(4\,\mathrm{e})) < \log(20m), \\
\Omega &= A_1A_2A_3 < 32\log(k)\log(2k)\log(1.55k) < 73.1(\log k)^3.
\end{aligned}$$

It follows from Theorem 4.2 that

$$\log|m\log\alpha_1 - n\log\alpha_2 + \log\alpha_3| > -2.3\cdot 10^{13}\log(20m)(\log k)^3.$$

Assume that $m \geq 10$. Since $\log(10\alpha_1^{-m'}) < -m' \log k$, we see from Lemma 4.1 that

$$2.3 \cdot 10^{13}(\log k)^2 > \frac{m'}{\log(20m)} =: F(m).$$

By Proposition 3.4, we have $F(m) < 2.3 \cdot 10^{13}(\log(511))^2 < 9 \cdot 10^{14}$. Since $F(m)$ is increasing and $F(10^{17}) > 10^{15}$, we obtain $m < 10^{17}$. □

Proof of Theorem 1.3. Dividing (4.1) by $\log \alpha_2$, we have

$$0 < m\kappa - n + \mu < AB^{-m}, \tag{4.2}$$

where

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2},\ \mu = \frac{\log \alpha_3}{\log \alpha_2},\ A = \frac{10}{\log \alpha_2},\ B = \alpha_2\ (k \geq 5)\ \text{ or }\ \alpha_2^{2/3}\ (k = 3).$$

Lemma 4.4. ([9, Lemma 5 a)]) *Let M be a positive integer and p/q a convergent of the continued fraction expansion of κ such that $q > 6M$. Put $\epsilon = ||\mu q|| - M||\kappa q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then the inequality* (4.2) *has no solution in the range*

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

As seen at the beginning in the proof of Proposition 3.4, we may assume that $n \geq 4$ for $z_0 = k$ and $n \geq 6$ for $z_0 = -k$. We can thus use Lemma 2.3, which implies $n \geq 2k^2 - 6$. We apply Lemma 4.4 with $M = 10^{17}$ for odd k with $3 \leq k \leq 511$ and for each sign of α_3. For $5 \leq k \leq 511$, the second convergent is needed in 2 cases and the third convergent is needed in two cases. In any case, the first step of reduction gives $m \leq 21$, which contradicts Lemma 2.3 with $k \geq 5$. For $k = 3$, the first step of reduction gives $m \leq 40$ and the second step gives $m \leq 10$ (the second convergent is needed only in the first step for the minus sign of α_3). This contradicts Lemma 2.3 with $k = 3$, and completes the proof of Theorem 1.3. □

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