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The number of primitive Pythagorean triples in a given interval

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Abstract: Integer structure analysis within the framework of modular rings is used to show that the formation of primitive (or "reduced") Pythagorean triples depends on certain characteristics with these rings. Only integers in the Class $\overline{1}_4$ of the modular ring Z_4 can produce primitive Pythagorean triples. Of these, a prime produces only one primitive Pythagorean triple, while composites produce the same number of primitive Pythagorean triples as their factors, provided the factors are square-free or are not elements of $\overline{3}_4$. Class $\overline{1}_4$ integers were converted to the equivalent Z_6 classes in order to isolate those divisible by 3. The numbers of primitive Pythagorean triples in various ranges were estimated and compared with the elder Lehmer's estimates. The results provide a neat link between the number of primitive Pythagorean triples and the number of primes in the given interval. It was also shown why the major component of a primitive Pythagorean triple is the only component which cannot have 3 as a factor.

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1 Introduction

Despite being studied over many centuries, Pythagorean triples still have aspects not yet fully explored. Modular rings and the analysis of their inherent integer structures are particularly suited to such studies of the simple equation

$$c^2 = a^2 + b^2 \tag{1.1}$$

We have recently shown [4] why primitive Pythagorean triples (pPts) always have one component with a factor of 5, while one of the two minor components always has a factor of 3. The reason the major component does not have a factor of 3 when the triple is primitive comes from the fact that only integers in Class $\overline{1}_4$ of the Modular Ring Z_4 (Table 1) can be a sum of squares: that is, $4r_1 + 1$ integers can equal a sum of squares whereas $4r_3 + 3$ cannot. Fermat proved this and it also follows from the integer structure. Only Classes $\overline{0}_4, \overline{1}_4$ contain even-powered integers. Thus, with b even and a odd: $c^2 \in \overline{1}_4$, $b^2 \in \overline{0}_4$, $a^2 \in \overline{1}_4$, [3]. Thus Equation (1.1) has the class structure

$$\overline{1}_4 = \overline{0}_4 + \overline{1}_4$$

so c must be in $\overline{1}_4$. Hence integers in Class $\overline{1}_4$ with 3 ($3 \in \overline{3}_4$) as a factor cannot form a pPt and hence c cannot have 3 as a factor.

To illustrate this more clearly and analyse how pPts are distributed, we convert from Z_4 to Z_6 (Table 2). The latter ring segregates odd integers with a factor of 3: all fall in Class $\overline{6}_6$ ($6r_3 + 3$, Table 2).

Row	***f*(*r*)**	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\overline{0}_4$	$\overline{1}_4$	$\overline{2}_4$	$\overline{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1. Rows of Z_4

Row	***f*(*r*)**	$6r_1-2$	$6r_2-1$	$6r_3$	$6r_4+1$	$6r_5+2$	$6r_6+3$
	Class	$\overline{1}_6$	$\overline{2}_6$	$\overline{3}_6$	$\overline{4}_6$	$\overline{5}_6$	$\overline{6}_6$
0		–2	–1	0	1	2	3
1		4	5	6	7	8	9
2		10	11	12	13	14	15
3		16	17	18	19	20	21
4		22	23	24	25	26	27
5		28	29	30	31	32	33
6		34	35	36	37	38	39
7		40	41	42	43	44	45

Table 2. Rows of Z_6

($\overline{2}_6$ and $\overline{5}_6$ have no even powers: $\left(\overline{2}_6\right)^m \in \overline{4}_6, \left(\overline{5}_6\right)^m \in \overline{1}_6$, m even)

The integers of Class $\overline{1}_4$ fall in $\overline{2}_6$ (odd rows), $\overline{4}_6$ (even rows) and $\overline{6}_6$ (odd rows). This can be verified from

$$c^2 = 4R_1 + 1$$

where

$$R_1 = 6(\tfrac{1}{2}n(3n \pm 1))$$

when $3 \nmid c$ if $3 \mid c$, then

$$R_1 = 2 + 18(\tfrac{1}{2}n(n+1)),\ \ n = 1, 2, 3, 4, \ldots$$

In the latter case, although c can be a sum of squares (usually when some factors are powers), the triple formed is not primitive as we see in the next section.

2 Distribution of primitive Pythagorean triples

The major component c is always in $\bar{1}_4$ and is given by

$$c = x^2 + y^2.$$

The minor components are formed from $2xy$ (even) and $|x^2 - y^2|$ (odd). Tables 3, 4 and 5 illustrate how the pPts are distributed for the range up to $c = 497$. The primes have only one set of (x, y) values. The other integers have the same number of (x, y) pairs as the number of factors, except for squares which have one pair. Obviously a square is not a prime, so that primes have distinct characteristics.

Integer	Factors	Factor Classes	x, y	Triples	pPts
5			2,1		5,4,3
17			4,1		17,8,15
29			2,5		29,20,21
41			4,5		41,40,9
53			2,7		53,28,45
65	5,13	$\bar{1}_4\ \bar{1}_4$	8,1		65,16,63
			4,7		65,56,33
77	7,11	$\bar{3}_4\ \bar{3}_4$	-		-
89			8,5		89,80,39
101			10,1		101,20,99
113			8,7		113,112,15
125	5,5,5	$\bar{1}_4\ \bar{1}_4\ \bar{1}_4$	2,11		125,44,117
			10,5	125,100,75	(5,4,3)
137			4,11		137,88,105
149			10,7		149,140,51
161	7,23	$\bar{3}_4\ \bar{3}_4$	-		-
173			2.13		173,52,165
185	5,37	$\bar{1}_4\ \bar{1}_4$	4,13 8,11		185,104,153 185,176,57
197			14,1		197,28,195

Integer	Factors	Factor Classes	*x*, *y*	Triples	pPts
209	11,19	$\overline{3}_4\ \overline{3}_4$	-		-
221	13,17	$\overline{1}_4\ \overline{1}_4$	14,5		221,140,171
			10,11		221,220,21
233			8,13		233,208,105
245	5,7,7	$\overline{1}_4\ \overline{3}_4\ \overline{3}_4$	14,7	245,196,147	(5,4,3)
257			16,1		257,32,255
269			10,13		269,260,69
281			16,5		281,160,231
293			2,17		293,68,285
305	5,61	$\overline{1}_4\ \overline{1}_4$	4,17		305,136,273
			16,7		305,224,207
317			14,11		317,308,75
329	7,47	$\overline{3}_4\ \overline{3}_4$	-		-
341	11,31	$\overline{3}_4\ \overline{3}_4$	-		-
353			8,17		353,272,225
365	5,73	$\overline{1}_4\ \overline{1}_4$	2,19		365,76,357
			14,13		365,364,27
377	13,29	$\overline{1}_4\ \overline{1}_4$	4,19		377,152,345
			16,11		377,352,135
389			10,17		389,340,189
401			20,1		401,40,399
413	7,59	$\overline{3}_4\ \overline{3}_4$	-		-
425	5,5,17	$\overline{1}_4\ \overline{1}_4\ \overline{1}_4$	8,19		425,304,297
			16,13		425,416,87
			20,5	425,200,375	(17,15,8)
437	19,23	$\overline{3}_4\ \overline{3}_4$	-		-
449			20,7		449,280.351
461			10,19		461,380,261
473	11,43	$\overline{3}_4\ \overline{3}_4$	-		-
485	5,97	$\overline{1}_4\ \overline{1}_4$	22,1		485,44,483
			14,17		485,476,93
497	7,71	$\overline{3}_4\ \overline{3}_4$	-		-

Table 3. Pythagorean Triples – Class $\overline{2}_6$ $(6r_2 - 1)$, r_2 odd

Integer	Factors	Factor Classes	*x*, *y*	Triples	pPts
13			2,3		13,12,5
25	5,5	$\bar{1}_4\ \bar{1}_4$	4,3		25,24,7
37			6,1		37,12,35
49	7,7	$\bar{3}_4\ \bar{3}_4$	-		-
61			6,5		61,60,11
73			8,3		73,48,55
85	5,17	$\bar{1}_4\ \bar{1}_4$	6,7		85,84,13
			2,9		85,36,77
97			4,9		97,72,65
109			10,3		109,60,91
121	11,11	$\bar{3}_4\ \bar{3}_4$	-		-
133	7,19	$\bar{3}_4\ \bar{3}_4$	-		-
145	5,29	$\bar{1}_4\ \bar{1}_4$	8,9		145,144,17
			12,1		145,24,143
157			6,11		157,132,85
169	13,13	$\bar{1}_4\ \bar{1}_4$	12,5		169,120,119
181			10,9		181,180,19
193			12,7		193,168,95
205	5,41	$\bar{1}_4\ \bar{1}_4$	6,13		205,156,133
			14,3		205,84,187
217	7,31	$\bar{3}_4\ \bar{3}_4$	-		-
229			2,15		229,60,221
241			4,15		241,120,209
253	11,23	$\bar{3}_4\ \bar{3}_4$	-		-
265	5,53	$\bar{1}_4\ \bar{1}_4$	12,11		265,264,23
			16,3		265,96,247
277			14,9		277,252,115
289	17,17	$\bar{1}_4\ \bar{1}_4$	8,15		289,240,161
301	7,43	$\bar{3}_4\ \bar{3}_4$	-		-
313			12,13		313.312.25
325	5,5,13	$\bar{1}_4\ \bar{1}_4\ \bar{1}_4$	18,1		325,36,323
			6,17		325,204,253
			10,15	325,300,125	(13,12,5)

Integer	Factors	Factor Classes	*x*, *y*	Triples	pPts
337			16,9		337,288,175
349			18,5		349,180,299
361	19,19	$\bar{3}_4\ \bar{3}_4$	-		-
373			18,7		373,252,275
385	5,7,11	$\bar{1}_4\ \bar{3}_4\ \bar{3}_4$	-		-
397			6,19		397,228,325
409			20,3		409,120,391
421			14,15		421,420,29
433			12,17		433,408,145
445	5,89	$\bar{1}_4\ \bar{1}_4$	18,11		445,396,203
			2,21		445,84,437
457			4,21		457,168,425
469	7,67	$\bar{3}_4\ \bar{3}_4$	-		-
481	13,37	$\bar{1}_4\ \bar{1}_4$	20,9		481,360,319
			16,15		481,480,31
493	17,29	$\bar{1}_4\ \bar{1}_4$	22,3		493,132,475
			18,13		493,468,155

Table 4: Pythagorean Triples – Class $\bar{4}_6$ $(6r_4 + 1)$, r_4 even

Integer	Factors	Factor Classes	*x*, *y*	Triples	pPts
9	3,3	$\bar{3}_4\ \bar{3}_4$			
21	3,7	$\bar{3}_4\ \bar{3}_4$			
33	3,11	$\bar{3}_4\ \bar{3}_4$			
45	3,3,5	$\bar{3}_4\ \bar{3}_4\ \bar{1}_4$	6,3	45,36,27	5,4,3
57	3,19	$\bar{3}_4\ \bar{3}_4$			
69	3,23	$\bar{3}_4\ \bar{3}_4$			
81	3,3,3,3,	$\bar{3}_4\ \bar{3}_4\ \bar{3}_4\ \bar{3}_4$			
93	3,31	$\bar{3}_4\ \bar{3}_4$			
105	5,3,7	$\bar{1}_4\ \bar{3}_4\ \bar{3}_4$			
117	3,3,13	$\bar{3}_4\ \bar{3}_4\ \bar{1}_4$			
129	3,43	$\bar{3}_4\ \bar{3}_4$			
141	3,47	$\bar{3}_4\ \bar{3}_4$			

Integer	Factors	Factor Classes	x, y	Triples	pPts
153	3,3,17	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$			
165	5,3,11	$\overline{1}_4\ \overline{3}_4\ \overline{3}_4$			
177	3,59	$\overline{3}_4\ \overline{3}_4$			
189	3,3,3,7	$\overline{3}_4\ \overline{3}_4\ \overline{3}_4\ \overline{3}_4$			
201	3,67	$\overline{3}_4\ \overline{3}_4$			
213	3,71	$\overline{3}_4\ \overline{3}_4$			
225	5,5,3,3	$\overline{1}_4\ \overline{1}_4\ \overline{3}_4\ \overline{3}_4$	12,9	225,216,63	(25,24,7)
237	3,79	$\overline{3}_4\ \overline{3}_4$			
249	3,83	$\overline{3}_4\ \overline{3}_4$			
261	3,3,29	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$	6,15	261,180,189	(29,20,21)
273	3,7,13	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$			
285	5,3,19	$\overline{1}_4\ \overline{3}_4\ \overline{3}_4$			
297	3,3,3,11	$\overline{3}_4\ \overline{3}_4\ \overline{3}_4\ \overline{3}_4$			
309	3,103	$\overline{3}_4\ \overline{3}_4$			
321	3,107	$\overline{3}_4\ \overline{3}_4$			
333	3,3,37	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$	18,3	333,108,315	(37,12,35)
345	5,3,23	$\overline{1}_4\ \overline{3}_4\ \overline{3}_4$			
357	3,7,17	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$			
369	3,3,41	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$	12,15	369,360,81	(41,40,9)
381	3,127	$\overline{3}_4\ \overline{3}_4$			
393	3,131	$\overline{3}_4\ \overline{3}_4$			
405	5,3,3,3,3	$\overline{1}_4\ \overline{3}_4\ \overline{3}_4\ \overline{3}_4\ \overline{3}_4$	18,9	405,324,243	(5,4,3)
417	3,139	$\overline{3}_4\ \overline{3}_4$			
429	3,11,13	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$			
441	3,3,7,7	$\overline{3}_4\ \overline{3}_4\ \overline{3}_4\ \overline{3}_4$			
453	3,151	$\overline{3}_4\ \overline{3}_4$			
477	3,3,53	$\overline{3}_4\ \overline{3}_4\ \overline{1}_4$	6,21	477,252,405	(53,28,45)
489	3,163	$\overline{3}_4\ \overline{3}_4$			

Table 5: Pythagorean Triples –Class $\overline{6}_6$ $(6r_3 + 1)$, r_3 odd

When the factors are in Class $\overline{3}_4$ no pPt is formed although (x, y) pairs appear with a common factor 3. Table 6 shows that the number of pPts is almost the same as the number of integers in a given range for $\overline{2}_6$ and $\overline{4}_6$ whereas $\overline{6}_6$ has no pPts since all integers there follow $3|c$.

For the range to 497, the final row in Z_4 equals 496 / 4 = 124. Thus there are 124 integers in this range (only one class being used for Z_4). For Z_6, 124 / 3 yields the approximate number of integers (Table 6) in a class. Since $\overline{6}_6$ has no pPts we can estimate that the number of pPts is around 83 (2 × 124 / 3), whereas it is 80.

Class	No. of integers	No. of triples	No. of pPts	No. of primes
$\overline{4}_6$	41	1	40	21
$\overline{2}_6$	42	3	40	23
$\overline{3}_6$	41	7	0	0
Total	124	11	80	44

Table 6: Number comparisons

Since the ratio of integers to pPts is constant for 100, 200, 300, 400 and 500 ranges, we can estimate that for 1000, R_1 = 250, so that the number of pPts is around 167 (2 × 250 / 3). In fact, the elder Lehmer proved that that the number of pPts with $c < X$ is approximately $(X / 2\pi)$ [2]. This yields 79.6 for X = 500 which compares favourably with the 80 found above. For X = 1000, the estimate is 159.2.

Another equation used by Lehmer to predict the number of pPts, M, is

$$M = (X \ln 2) / \pi^2$$

in which X is the perimeter of the last pPt in the range. For example, with c = 997 (a prime), the perimeter of the pPt is (997 + 372 + 925) = 2294, so that M = 161, which is similar to the result from the previous formula. Our rough estimate tends to yield a number which is about 4% higher.

3 Concluding comments

The primes (Table 6) contribute more than half the total. This provides a link between the number of primes (in $\overline{1}_4$) and the pPts in the range.

Of relevance too is that Hall [1] proved that (a, b, c) from Equation (1.1) is a pPt iff

$$(a, b, c) = (3, 4, 5)\, M$$

in which M is a finite product of the unimodular matrices U, A, D defined by

$$U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \; D = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

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