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A short note on the Inclusion-Exclusion principle: A modification with applications

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Abstract: The Inclusion-Exclusion (I.E.) principle is an important counting concept in combinatorics. It is also very important in the study of the distribution of prime numbers. In this paper, we introduce an equivalent - and in some cases a relatively easier to apply - form of the concept. We also provide some applications.
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1 Introduction

Given a set X, let $|X|$ be the cardinality of X. Throughout this paper, the set X is non-empty, unless otherwise stated. Suppose $A_i \subseteq X, (1 \leq i \leq n; i, n \in \mathbb{N})$ are distinct subsets of X. We can consider the elements of A_i as those elements of X that satisfy some given property $K(i)$. Naturally, one may want to know the value of

$$\left|\bigcup_{i=1}^{n} A_i\right|. \tag{1}$$

If we fall on the Inclusion-Exclusion principle, we have

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}(-1)^{k+1} \sum_{1\leq i_1<\ldots<i_k\leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|. \tag{2}$$

It is clear from the right-hand side of the equation in (2) that the I.E. principle is useful if we have an easy way of evaluating the value of

$$\sum_{i=1}^{n}(-1)^{k+1} \sum_{1\leq i_1<\ldots<i_k\leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|. \tag{3}$$

Unfortunately, this is not possible all the time, and in pratice we are usually forced to resort to approximation techniques that allow us to estimate the value of $|\cup_{i=1}^{n} A_i|$ instead. Traditionally, we are interested in upper or lower bounds of $|\cup_{i=1}^{n} A_i|$ and even in this case, the results we get are not really what we desired at first. There are many versions of the I.E. principle available but all of them suffer from the existence of the object $(-1)^k$, where k is a non-negative integer. The presence of $(-1)^k$ in the right-hand side of (2) means that some background cancellation must take place for the equality to work. This is the main reason why it is difficult to apply the identity(2) in many instances where the value of n is large. For small values of n, there are not many $+(s)$ and $-(s)$ to keep track of and so the I.E. principle works very well.

The purpose of this paper is the introduction of a form of the I.E. principle that gets around the problem associated with managing the background cancellations associated with (2). We begin with some basic definitions and notations. The set $\mathbb{P}$ shall denote the set of prime numbers and p is reserved for primes. We shall also use

$$[m] = \{1, 2, 3, \ldots, m\}$$

and

$$S_m = \{f : [m] \to [m] | f \text{ is a bijection}\},$$

where m is a positive integer. Now, a little modification to the equation in (2) gives

$$\left| X \backslash \bigcup_{i=1}^{n} A_i \right| = |X| - \sum_{i=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|. \tag{4}$$

That is, $|X \backslash \cup_{i=1}^{n} A_i|$ is the number of elements of X that do not belong to $|\cup_{i=1}^{n} A_i|$. In the next section, we will provide an alternative way to evaluate $|\cup_{i=1}^{n} A_i|$.

1.1 Another way to evaluate $|\cup_{i=1}^{n} A_i|$

Let $\sigma \in S_m$, where $m \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n$ distinct subsets of X. Define

$$N_X^G(\sigma(1)) = \{y \in X | y \in A_{\sigma(1)}\},$$

$$N_X^G(\sigma(1), \sigma(2)) = \{y \in X \backslash N_X^G(\sigma(1)) | y \in A_{\sigma(2)}\}$$

and in general

$$N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(r)) = \{y \in X \backslash \bigcup_{j=1}^{r-1} N_X^G(\sigma(1), \ldots, \sigma(j)) | y \in A_{\sigma(r)}\};$$

for each $r (1 \le r \le n)$, where $G = \{A_1, A_2, \ldots, A_n\}$. In most cases where we use G without explanation, one should deduce the meaning from the context. With these definitions, we have the following result.

Lemma 1. *Let $A_1, A_2, \ldots, A_n$ be distinct subsets of some non-empty set X. The following statements holds for each $\sigma \in S_n, n \in \mathbb{N}$.*
(i)

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i)),$$

(ii)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \left|\bigcup_{i=1}^{n} N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))\right|,$$

where $G = \{A_1, A_2, \ldots, A_n\}$.

Proof. (i) We prove the case $n = 2$.
Claim: $A_1 \cup A_2 = N_X^G(\sigma(1)) \cup N_X^G(\sigma(1), \sigma(2))$. This is easy, since $N_X^G(\sigma(1))$ is the set of elements of $A_1 \cup A_2$ that belongs to $A_{\sigma(1)}$ and $N_X^G(\sigma(1), \sigma(2))$ is the set of elements of $A_1 \cup A_2$ that belongs to $(A_1 \cup A_2)\backslash A_{\sigma(1)}$ and $A_{\sigma(2)}$. But

$$(A_1 \cup A_2)\backslash A_{\sigma(1)} = ((A_1 \cup A_2)\backslash A_{\sigma(1)}) \cap A_{\sigma(2)}$$

and so the result follows. The general case follows easily by applying the inductive process on n.
(ii) This follows directly from the statement in (i). □

The statement in Lemma 1 is equivalent - in a way - to the I.E. principle. However, since we do not have to worry about double-counting, one can refer to Lemma 1 as the Exclusion principle. The following equality is easy to astablish.

$$\left|X\backslash\bigcup_{i=1}^{n} A_i\right| = |X| - \left|\bigcup_{i=1}^{n} N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))\right| = |X| - \sum_{i=1}^{n} |N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))|. \tag{5}$$

It is easy to see that

$$\left|\bigcup_{i=1}^{n} N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))\right| = \sum_{i=1}^{n} |N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))|$$

is 'much easier' to work with than

$$\sum_{i=1}^{n}(-1)^{k+1} \sum_{1 \le i_1 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|,$$

even though in most cases, it is generally difficult to determine the exact values of both quantities. We present a simple example to verify the 'Exclusion-Principle'.

Example 2. *Let $X = \{41, 42, 43, \ldots, 50\}$, $A = \{x \in X : 2|x\}$ and $B = \{x \in X : 3|x\}$. Set $G = \{A, B\}$, then*

$$A \cup B = N_X^G(A) \cup N_X^G(A, B) = N_X^G(B) \cup N_X^G(B, A).$$

We have $N_X^G(A) = \{42, 44, 46, 48, 50\}$ *and* $N_X^G(A, B) = \{45\}$ *and so*

$$N_X^G(A) \cup N_X^G(A, B) = \{42, 44, 45, 46, 48, 50\}.$$

Similarly, $N_X^G(B) = \{42, 45, 48\}$ *and* $N_X^G(B, A) = \{44, 46, 50\}$ *and so*

$$N_X^G(B) \cup N_X^G(B, A) = \{42, 44, 45, 46, 48, 50\}.$$

In the next section, we will use the I.E. principle to provide alternative proofs to some number theoritical problems.

2 Application to the distribution of prime numbers

Let $x \in \mathbb{N}$ and define $X(x) = [1, x] \cap \mathbb{N} = \{1, 2, 3, \ldots, x\}$ for each $x \in \mathbb{N}$ and

$$A = \{p_1, p_2, \ldots, p_n | p_1 < p_2 < \cdots < p_n | p_i, p_2, p_3, \ldots, p_n \in \mathbb{P}\}.$$

Define

$$S_*(A, X(x)) = |\{a \in X(x) | \gcd(a, \prod_{p \in A} p) > 1\}|. \tag{6}$$

We have

$$S_*(A, X(x)) = \sum_{i=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left| A_{i_1}^{X(x)} \cap A_{i_2}^{X(x)} \cap \cdots \cap A_{i_k}^{X(x)} \right|, \tag{7}$$

where $A_{i_1}^{X(x)} \cap A_{i_2}^{X(x)} \cap \cdots \cap A_{i_k}^{X(x)} = \{y \in X(x) | \gcd(y, p_{i_j}) > 1, \forall j (1 \leq j \leq k)\}$, for each k. In the setting above, we chose to avoid the Möbius function. Related to the equality in (7) is the following statement;

$$S(A, X(x)) = |X(x)| - \sum_{i=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left| A_{i_1}^{X(x)} \cap A_{i_2}^{X(x)} \cap \cdots \cap A_{i_k}^{X(x)} \right|, \tag{8}$$

In practice, it is natural to assume that

$$\left| A_{i_1}^{X(x)} \cap A_{i_2}^{X(x)} \cap \cdots \cap A_{i_k}^{X(x)} \right| = g(i_1, i_2, \ldots, i_k) |X(x)| + Error(i_1, i_2, \ldots, i_k) \tag{9}$$

for some function $g : 0 \leq g(i_1, i_2, \ldots, i_k) \leq 1$ and then develop techniques for managing the cummulative error created when we substitute the expression in (9) into the equations in (7) or (8). Various methods in sieve theory have been developed as a result. We will deal with this problem in a slightly different way – using the Exclusion principle. We proceed as follows. Let $\sigma \in S_n$ and $A, X(x)$; sets defined in the beginning of this section. Set

$$A_i = \{y \in X(x) : p_i | y\}, G = \{A_1, A_2, \ldots, A_n\}$$

$$U_{X(x)}^G(\sigma(1)) = \left\lceil \frac{|X(x)|}{p_{\sigma(1)}} \right\rceil$$

and

$$U_{X(x)}^G(\sigma(1),\sigma(2),\ldots,\sigma(k)) = \left\lceil \frac{|X(x)| - \sum_{i=1}^{k-1} U_{X(x)}^G(\sigma(1),\sigma(2),\ldots,\sigma(r))}{p_{\sigma(k)}} \right\rceil, \tag{10}$$

$\forall k (1 \leq k \leq n)$. Furthermore, we have

$$U_{X(x)}^G[\sigma] = \sum_{s=1}^{n} U_{X(x)}^G(\sigma(1),\ldots,\sigma(s)), \tag{11}$$

for each $\sigma \in S_n$. The expression in (11) varies with $\sigma \in S_n$ and so it is natural to define

$$U(G, X(x)) = \max\{U_{X(x)}^G[\sigma] : \sigma \in S_n\}. \tag{12}$$

We present an example to shed additional light on the definitions above.

Example 3. *Let $X(10) = \{1,2,3,\ldots,10\}$ and $A = \{2,3,5\}$, then we have*
(i) $\sigma = (2,3,5)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(2) + U_{X(10)}^G(2,3) + U_{X(10)}^G(2,3,5) = \left\lceil \frac{10}{2} \right\rceil + \left\lceil \frac{5}{3} \right\rceil + \left\lceil \frac{3}{5} \right\rceil = 5+2+1 = 8.$$

(ii) $\sigma = (2,5,3)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(2) + U_{X(10)}^G(2,5) + U_{X(10)}^G(2,5,3) = \left\lceil \frac{10}{2} \right\rceil + \left\lceil \frac{5}{5} \right\rceil + \left\lceil \frac{4}{3} \right\rceil = 5+1+2 = 8.$$

(iii) $\sigma = (3,2,5)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(3) + U_{X(10)}^G(3,2) + U_{X(10)}^G(3,2,5) = \left\lceil \frac{10}{3} \right\rceil + \left\lceil \frac{6}{2} \right\rceil + \left\lceil \frac{3}{5} \right\rceil = 4+3+1 = 8.$$

(iv) $\sigma = (3,5,2)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(3) + U_{X(10)}^G(3,5) + U_{X(10)}^G(3,5,2) = \left\lceil \frac{10}{3} \right\rceil + \left\lceil \frac{6}{5} \right\rceil + \left\lceil \frac{4}{2} \right\rceil = 4+2+2 = 8.$$

(v) $\sigma = (5,2,3)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(5) + U_{X(10)}^G(5,2) + U_{X(10)}^G(5,2,3) = \left\lceil \frac{10}{5} \right\rceil + \left\lceil \frac{8}{2} \right\rceil + \left\lceil \frac{4}{3} \right\rceil = 2+4+2 = 8.$$

(vi) $\sigma = (5,3,2)$;

$$U_{X(10)}^G[\sigma] = U_{X(10)}^G(5) + U_{X(10)}^G(5,3) + U_{X(10)}^G(5,3,2) = \left\lceil \frac{10}{5} \right\rceil + \left\lceil \frac{8}{3} \right\rceil + \left\lceil \frac{5}{2} \right\rceil = 2+3+2 = 7.$$

Therefore, we have $U(G, X(10)) = 8$, where $G = \{A_2^{X(10)}, A_3^{X(10)}, A_5^{X(10)}\}$ and

$$A_p^{X(10)} = \{y \in X(10) : p|y\}, p \in A.$$

Notice that, using the I.E. principle, we should have

$$S_*(A, X(10)) = 8.$$

That is, $U(G, X(10)) \geq S_*(A, X(10))$. This statement is true in general.

Theorem 4. *Let* $X(x) = \{1, 2, 3, \ldots, x\}$ *for some* $x \in \mathbb{N}$*;* $A = \{p_1, p_2, \ldots, p_n | p_1 < p_2 < \cdots p_1, p_2, \ldots, p_n \in \mathbb{P}\}$ *and* $G = \{A_{p_i}^{X(x)} : 1 \leq i \leq n\}$*, then* $U(G, X(x)) \geq S_*(A, X(x))$*.*

The following statement is aslo important in measuring $U(G, X(x))$.

Lemma 5. *Let* $X(x) = \{1, 2, 3, \ldots, x\}$ *for some* $x \in \mathbb{N}$*;* $A = \{p_1, p_2, \ldots, p_n | p_1 < p_2 < \cdots p_1, p_2, \ldots, p_n \in \mathbb{P}\}$ *then* $U(G, X(x)) \leq U^G_{X(x)}[id]$ $id : [n] \to [n]$ *is the identity permutation.*

Proof. Assume the definition of $X(x)$, G and A. Suppose

$$U_{X(x)}G[\sigma] \leq U^G_{X(x)}[id]$$

for all $\sigma \in S_n$. $p_{id(i)} \leq p_{\sigma(i)}, \forall i$ and so $U^G_{X(x)}[id]$ is more sensitive (first to increase, term by term) to changes in x than $U^G_{X(x)}[\sigma]$. Therefore, for $r \in \mathbb{N}$, we should have $U^G_{X(x+r)}[id] \geq U^G_{X(x+r)}[\sigma]$ and this should be the case for any n. $\square$

Example 6. *Suppose we want to find an upper bound for* $S_*(\{2, 5, 7\}, X(20))$*. Since* $2, 5$ *and* 7 *are all less than* 20*, we have* $U^G_{X(20)}[(2, 5, 7)]$ *as a fairly good estimate. In fact,*

$$U^G_{X(20)}[(2, 5, 7)] = \left\lceil \frac{20}{2} \right\rceil + \left\lceil \frac{10}{5} \right\rceil + \left\lceil \frac{8}{7} \right\rceil = 10 + 2 + 2 = 14.$$

The actual value is 13*.*

If we replace the ceiling function with the floor function in (10), we get

$$L^G_{X(x)}(\sigma(1)) = \left\lfloor \frac{|X(x)|}{p_{\sigma(1)}} \right\rfloor$$

and

$$L^G_{X(x)}(\sigma(1), \sigma(2), \ldots, \sigma(k)) = \left\lfloor \frac{|X(x)| - \sum_{i=1}^{k-1} L^G_{X(x)}(\sigma(1), \sigma(2), \ldots, \sigma(r))}{p_{\sigma(k)}} \right\rfloor, \tag{13}$$

$\forall k(1 \leq k \leq n)$. Furthermore, we have

$$L^G_{X(x)}[\sigma] = \sum_{s=1}^{n} L^G_{X(x)}(\sigma(1), \ldots, \sigma(s)), \tag{14}$$

for each $\sigma \in S_n$. The expression in (14) varies with $\sigma \in S_n$ and so it is natural to define

$$L(G, X(x)) = \min\{L^G_{X(x)}[\sigma] : \sigma \in S_n\}. \tag{15}$$

With these definitions, it is easy to see that the following result should hold.

Theorem 7. *Let* $X(x) = \{1, 2, 3, \ldots, x\}$ *for some* $x \in \mathbb{N}$*;* $A = \{p_1, p_2, \ldots, p_n | p_1 < p_2 < \cdots p_1, p_2, \ldots, p_n \in \mathbb{P}\}$ *and* $G = \{A_{p_i}^{X(x)} : 1 \leq i \leq n\}$*, then*

$$L(G, X(x)) \leq S_*(A, X(x)) \leq U(G, X(x)).$$

The inequality in this theorem is weak but useful.

2.1 Some results

The definition of $U^G_{X(x)}[id]$ depends largely on the ceiling function. In order to use it to prove some statements in prime number theory, we have to provide some basic results which will be useful later.

Lemma 8. *For each $x > 0, n \in \mathbb{N}$, we have $\lceil nx \rceil \le n\lceil x \rceil$.*

Proof. Let $x > 0, n \in \mathbb{N}$. We have $\lceil nx \rceil = \lceil x \rceil + \left\lceil x - \frac{1}{n} \right\rceil + \left\lceil x - \frac{2}{n} \right\rceil + \cdots + \left\lceil x - \frac{n-1}{n} \right\rceil \le \sum_{i=1}^{n} \lceil x \rceil = n\lceil x \rceil$. □

The equality

$$\lceil nx \rceil = \lceil x \rceil + \left\lceil x - \frac{1}{n} \right\rceil + \left\lceil x - \frac{2}{n} \right\rceil + \cdots + \left\lceil x - \frac{n-1}{n} \right\rceil$$

is called Hermite's identity.

Lemma 9. *For each $\sigma \in S_n, n \in \mathbb{N}$, we have*

$$U^G_{X(x)}[\sigma] \le U^G_{X(x+r)}[\sigma]$$

for all $r, x \in \mathbb{N}$, (G fixed) and

$$U^G_{X(rx)}[id] \le rU^G_{X(x)}[id].$$

Proof. The first inequality follows easily since the components of σ are fixed and $r > 0$. The second inequality follows by applying Lemma 7. □

From Lemma 8, we have $U(G, X(x+r)) \ge U(G, X(x))$ in general, where G has its usual meaning. Now, let $x \in \mathbb{N}, x > 1$ and $\mathbb{P}(x)$ the set of prime numbers less than or equal to x. We have $S_*(\mathbb{P}(x), X(x)) = x - 1$ since $\gcd(1, p) = 1, \forall p \in \mathbb{P}(x)$. To show that there are infinitely-many prime numbers, one must show that there exists some $y > x$ such that $S_*(\mathbb{P}(x), X(y)) < y - 1$. Alternatively, one can show that $|S_*(\mathbb{P}(x), X(x+r)) - (x+r)| \to \infty$ as $r \to \infty$. We shall now prove that there are infinitely-many prime numbers. The basic idea is applicable to other problems concerning the distribution of primes.

Theorem 10. *There are infinitely-many prime numbers.*

Proof. Suppose there exists $x \in \mathbb{N}$ such that $\mathbb{P}(x) = \mathbb{P}$, then we should have

$$x - 1 = S_*(\mathbb{P}(x), X(x)) \le U^G_{X(x)}[id], \tag{16}$$

where $G = \{A_p^{X(x)}(x) : p \in \mathbb{P}(x)\}$ and $A_p^{X(x)}(x) = \{z \in X(x) : p|z\}$. By assumption, one should have

$$|U^G_{X(kx)}[id] - kx| \le 1$$

for each $k > 1$. However, $|U^G_{X(kx)}[id] - kx| \to \infty$ as $k \to \infty$. Therefore, $\mathbb{P}(x) = \mathbb{P}$ is not possible. There must be a prime number greater than x. □

The following result is also useful.

Lemma 11. *For primes $p_1 < p_2 < \cdots < p_n$, we have*

$$\sum_{i=1}^{n} U_{X(x)}^{G}(p_1, p_2, \ldots, p_i) \leq \sum_{k=2}^{n+1} U_{X(x)}^{S}(2, 3, \ldots, k),$$

where $G = \{A_{p_i}^{X(x)}(x) : p_1 < \cdots < p_n\}, S = \{A_J^{X(x)}(x) : 2 \leq J \leq n+1\}$.

Therefore, if

$$\sum_{k=2}^{n+1} U_{X(x)}^{S}(2, 3, \ldots, k) < x - 1,$$

we can expect the interval $[1, x]$ to contain at least n primes. If $2 = p_1 < p_2 < \cdots < p_n < \cdots$ is the sequence of prime numbers and $X(x) = \{1, 2, 3, \ldots, x\}$, the expression $\sum_{i=1}^{n} U_{X(x)}^{G}(p_1, p_2, \ldots, p_n)$ can provide us with some insight into the distribution of prime numbers. For example, if

$$\sum_{i=1}^{n} U_{X(p_n)}^{G}(p_1, p_2, \ldots, p_i) \geq p_n - 1,$$

for some n then there exists $r \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} U_{X(p_n+r)}^{G}(p_1, p_2, \ldots, p_i) < p_n + r - 1.$$

This means that there must be some prime number in the interval $(p_n, p_n + r]$.

3 Conclusion

The equality

$$\left|\bigcup_{i=1}^{n} A_i\right| = \left|\bigcup_{i=1}^{n} N_X^G(\sigma(1), \sigma(2), \ldots, \sigma(i))\right|,$$

where $G = \{A_1, A_2, \ldots, A_n\}$ is a nice way of looking at the I.E. principle and there might be some interesting results and applications one might find later one. For example, there appear to some useful application to the following statements. Chebychev's theorem concerning the distribution of prime numbers within the interval $(n, 2n], n \in \mathbb{N}$ was first proved by P.L. Chebyschev and simplified later by P. Erdós [6]. M. El. Bachraoui [4] extended Chebychev's theorem to the interval $(2n, 3n]$. Unfortunately, it is unclear how one can extend their results, using their methods, to the interval $(kn, (k+1)n]$ for all $n, k \in \mathbb{N}, (1 \leq k \leq n+1)$.
Cram$\acute{e}$r [1] conjectured that every interval $(n, n + f(n)log^2 n)$ contains a prime for some $f(n) \to 1$ as $n \to \infty$ and Harman [3] proved, using sieve methods, that for almost all n, the interval $(n, n + n^{\frac{1}{10}+e}), e > 0$ contains a prime number.

Andrica Dorin [2] conjectured that the inequality

$$p_{n+1} \leq p_n + 2\sqrt{p_n} + 1, \forall n \geq 1 \tag{17}$$

holds for all $n \geq 1$. This inequality in (17) is checked, computationally, for n up to 1.3×10^6. Equally important is the conjecture proposed by Oppermann [5] which states that for each $x > 1$, there exists prime numbers p, r satisfying $x(x-1) < p < x^2$ and $x^2 < r < x(x+1)$.

All the conjectures we have presented concern the distribution of prime numbers in short intervals and this area of research is very important to Number Theorists and Mathematicians in general. Using some properties of the Exclusion-principle and permutations, there appear to be some reason to believe that we can say something positive about the conjectures listed above. We are still studying possible connections.

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