Quotients of primes in arithmetic progressions

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Abstract: We prove an open problem of Hobby and Silberger on quotients of primes in arithmetic progressions.

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Let \( a \) and \( b \) be positive coprime integers and denote by \( D(a, b) \) the set of prime numbers which are congruent to \( b \) modulo \( a \). For a given set \( S \subseteq \mathbb{N} \) write \( F(S) \) for the set of all quotients of elements of \( S \).

In 1993, Hobby and Silberger [1] proved that if \( \mathbb{P} \) is the set of all prime numbers, then \( F(\mathbb{P}) \) is dense in \( \mathbb{R}^+ := [0, \infty) \). As an open problem they asked for the generalization of this result to arithmetic progressions; that is, decide whether \( F(D(a, b)) \) is dense in \( \mathbb{R}^+ \). Two years later, Starni [2] claimed to answer Hobby and Silberger’s question in the affirmative, though it seems that their proof has a flaw. Indeed, if we write \( p_{a, b}(n) \) for the \( n \)-th prime in \( D(a, b) \), Starni claimed that \( p_{a, b}(n) \sim n \log n \) which is false as we shall prove in the following lemma.

Lemma. If \( p_{a, b}(n) \) is as defined above, then \( p_{a, b}(n) \sim \phi(a)n \log n \), where \( \phi(q) \) is Euler’s totient function.

Proof. Denote by \( \pi(x; a, b) \) the number of primes up to \( x \) that are congruent to \( b \) modulo \( a \). The prime number theorem for arithmetic progressions thus implies that

\[
\lim_{n \to \infty} \frac{\phi(a)n \log p_{a, b}(n)}{p_{a, b}(n)} = 1.
\]

Taking logarithms and dividing by \( \log p_{a, b}(n) \) we obtain

\[
\lim_{n \to \infty} \frac{\log n}{\log p_{a, b}(n)} = 1.
\]

Multiplying the two above limits together proves the lemma. \( \square \)
Note that Starni’s claim with this lemma, Starni’s prove goes through nicely. To make this paper self contained and a bit more interesting, we use the above lemma to give a simpler proof of the density of $F(D(a, b))$ is dense in $\mathbb{R}^+$. 

**Theorem.** We have $F(D(a, b))$ is dense in $\mathbb{R}^+$. 

**Proof.** Let $[y]$ denote the integer part of the positive real number $y$. By the above lemma we have for a given real number $x \in \mathbb{R}^+$ that 

$$
\lim_{n \to \infty} \frac{p_{a,b}([xn])}{p_{a,b}(n)} = x. \quad \square
$$

We note that this type of argument was suggested by M. Mendès France in his review of [1].

**References**
