Short remarks on Jacobsthal numbers

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
IBPhBME – Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Abstract: Some new generalization of the Jacobsthal numbers are introduced and properties of the new number are studied.

Keywords: Fibonacci number, Jacobsthal number, Recurrence.

AMS Classification: 11B37.

The \( n \)-th Jacobsthal number \((n \geq 0)\) is defined by

\[
J_n = \frac{2^n - (-1)^n}{3}
\]  \hspace{1cm} (1)

(see, e.g., [1]). In [2], it is generalized to the form

\[
J^s_n = \frac{s^n - (-1)^n}{s + 1},
\]  \hspace{1cm} (2)

where \( n \geq 0 \) is a natural number and \( s \geq 0 \) is a real number.

The presumption of this generalization is that number 2 is changed with \( s \) and therefore, 3 must be changed with \( s + 1 \).

Here another generalization is introduced, interpreting 3 not as the next number after 2, but as \( 2^2 - 1 \), i.e., changing it by \( s^2 - 1 \). In a result, the following new numbers are obtained

\[
Y^s_n = \frac{s^n - (-1)^n}{s^2 - 1},
\]  \hspace{1cm} (3)

where \( s \neq 1 \) is a real number.

Obviously, when \( s = 2 \) we obtain the standard Jacobsthal numbers.

In the case \( s = 0 \), we obtain

\[
Y^0_n = (-1)^n.
\]

The first six members of the sequence \( \{Y^s_n\} \) with respect to \( n \) are

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>( \frac{1}{s-1} )</td>
<td>( 1 )</td>
<td>( s + \frac{1}{s-1} )</td>
<td>( s^2 + s + \frac{1}{s-1} )</td>
<td>( s^3 + s + \frac{1}{s-1} )</td>
<td></td>
</tr>
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</table>
It can be directly seen from (2) and (3) that for the real number $s \neq 1$
\[ Y_n^s = \frac{1}{s-1}J_n^s. \]

**Theorem 1.** For every natural number $n \geq 0$ and real number $s \neq 1$:
\[ Y_{n+2}^s = Y_n^s + s^n, \]
\[ Y_{n+1}^s \equiv s.Y_n^s + \frac{(-1)^n}{s-1}. \]

**Proof.** It can be directly checked that for each $n \geq 0$:
\[ Y_{n+2}^s - Y_n^s = \frac{1}{s^2-1}(s^{n+2} - (-1)^{n+2} - s^n + (-1)^n) = s^n. \]
\[ Y_{n+1}^s - s.Y_n^s = \frac{1}{s^2-1}.(s^{n+1} - (-1)^{n+1} - s^{n+1} + (-1)^n.s) = \frac{(-1)^n}{s-1}. \]

Two next steps of modification of the Jacobsthal numbers have the following forms.
First, we can mention that 2 and 3 are the first two prime numbers and therefore, the Jacobsthal numbers can obtain the form
\[ JP_n^s = \frac{p_n^s - (-1)^n}{p_{n+1}}, \quad (4) \]
where $p_i$ is the $i$-th prime number ($p_0 = 2, p_1 = 3, ...$).
Second, we can mention that 2 and 3 are two sequential Fibonacci numbers and, therefore, the Jacobsthal numbers can obtain the form
\[ JF_n^s = \frac{f_n^s - (-1)^n}{f_{n+1}}, \quad (5) \]
where $f_i$ is the $i$-th Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, ...$).
For the latest numbers we see that the following assertion is valid.

**Theorem 2.** For every natural number $n \geq 0$ and real number $s \neq 1$:
\[ JF_{n+1}^s = JF_n^s + s^n + \frac{f_n - 1}{f_{n+1}}.f_n. \]

An open problem is to study the properties of numbers $JP_n^s$ and $JF_n^s$.

**References**
