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Odd-powered triples and Pellian sequences

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Abstract: Even and odd integers of the form N^m in the modular ring Z_4 have rows with elements which satisfy Pellian recurrence relations from which Pythagorean triples can form when $m = 2$. When m is odd, they have different and incompatible Pellian row structure and triples are not formed.

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1 Introduction

The most important feature of Integer Structure Analysis (ISA) is the multifarious character of the structure of integers [3]. For instance, when the right-end-digit (RED) patterns or algebraic equations are involved, Z_5 is the preferred modular ring because classes are RED specific and of mixed parity. The ring Z_6 is preferred if divisibility or non-divisibility by 3 needs to be structurally distinguished as integers divisible by 3 occur in one class for each parity [3].

On the other hand, Z_4 is particularly useful for analysis of the structure of π [8] and of power functions [5, 9], such as why Pythagorean triples always have factors of 3 in one of the minor components and 5 in one of the components [4, 6].

In this paper, we illustrate how the elements of rows of powers belong to different, and apparently conflicting, sequences. In particular, we analyse some aspects of Pythagorean triples.

2 Rows of squares

The structure of the Pythagorean equation in Z_4 (Table 1)

$$c^2 = a^2 + b^2 \tag{2.1}$$

may be summarised by:

- $c \in \overline{1}_4$ as this is the only Class to contain odd integers equal to a sum of squares;
- one of the minor components, a or b, always has a factor of 3;
- one of the components, a, b or c, always has a factor of 5;
- the elements of the rows of the squares of the odd components are members of the sequence $\{6D_n\}$ (where D_n are the pentagonal numbers) if the elements are not divisible by 3;
- the elements of the rows of the squares of the odd components are members of the sequence $\{2 + 18T_n\}$ (where T_n are the triangular numbers) if the elements are divisible by 3;
- the elements of the rows of squares in general are members of Pellian-type sequences irrespective of whether the elements are divisible by 3 or not;
- the even components cannot belong to $\overline{2}_4$.

Row	***f*(*r*)**	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\overline{0}_4$	$\overline{1}_4$	$\overline{2}_4$	$\overline{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1. Rows of Z_4

Odd squares always belong to the Class $\overline{1}_4$ and the elements of the rows satisfy a non-homogeneous form of a Pellian-type recurrence relation (2.2) with t = 2 and with suitable initial conditions (Table 2):

$$R_{i+1} = 2R_i - R_{i-1} + t. \tag{2.2}$$

N	1	3	5	7	9	11	13	15	17
$\boldsymbol{N^2}$	1	9	25	49	81	121	169	225	289
Row	0	2	6	12	20	30	42	56	72
i	0	1	2	3	4	5	6	7	8

Table 2. Rows of odd squares

Even squares $N^2 \in \overline{0}_4$ with $N \in \overline{0}_4$ only. The elements of these rows satisfy (2.2) with t = 8. Thus, with $c^2 = 4R_1 + 1$, $b^2 = 4R_0$, $a^2 = 4R_1^{/} + 1$, Equation (2.1) can be re-written as

$$R_1 = R_0 + R_1^{/}. \tag{2.3}$$

Obviously the rows are compatible, which allows primitive Pythagorean triples (pPts) to form:

$$2R_k - R_{k-1} = 2(R_i + R_j + 4) - (R_{i-1} + R_{j-1}). \tag{2.4}$$

For example, for pPt (24, 7, 25):

$$R_i = 6, \quad R_j = 100, \quad R_k = 132,$$
$$R_{i-1} = 2, \quad R_{j-1} = 64, \quad R_{k-1} = 110,$$

and Equation (2.4) is then: 264 – 110 = 154 = 2(6 + 100 + 4) – (2 + 64).

3 Rows of cubes

We have seen the rows in relation to triangular and pentagonal numbers, but these do not display the incompatibility of these rows for triples. We shall look further at Pellian-type sequences with the aid of Tables 3 and 4 which give examples of the row structures in Classes $\bar{1}_4$ and $\bar{0}_4$, respectively.

N	1	5	9	13	17	21	25	29
N^3	1	125	729	2197	4913	9261	15625	24389
Row	0	31	182	549	1228	2315	3906	6097
i	0	1	2	3	4	5	6	7

Table 3. Cubes in Class $\bar{1}_4$

N	2	4	6	8	10	12	14	16	18
N^3	8	64	216	512	1000	1728	2744	4096	5832
Row	2	16	54	128	250	432	686	1024	1458
i	0	1	2	3	4	5	6	7	8

Table 4. Even cubes $\in \bar{0}_4$

A note which is not totally irrelevant here since it leads to related research with cubes is that 1728 is related to the Hardy-Ramanujan number, $Ta(2) = 1728 + 1$. This is famous for the story that when Hardy visited the ailing Ramanujan on one occasion he observed by way of starting the conversation that he had arrived in a cab numbered 1729 which seemed to be uninteresting. Ramanujan immediately stated that it was actually a very interesting number mathematically, being the smallest natural number representable in two different ways as a sum of two different cubes [1]:

$$Ta(2) = 1^3 + 12^3 = 9^3 + 10^3;$$

these "taxicab numbers", $Ta(n)$, are now defined as the smallest number that can be expressed as a sum of two positive algebraic cubes in n distinct ways. Thus,

$$Ta(1) = 2 = 1^3 + 1^3,$$
$$Ta(2) = 1729 = 1^3 + 12^3 = 9^3 + 10^3,$$
$$Ta(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3.$$

Hardy and Wright [2] proved that such numbers exist for all positive integers *n*, but their proof makes no claims at all about whether the numbers they generate are the smallest possible and thus it cannot be used to find the actual value of *Ta*(*n*).

We return to the Tables 3 and 4 and observe that for $\bar{1}_4$ the rows of the cube are given by the second order homogeneous recurrence relation

$$R_{i+1} = 2R_i - R_{i-1} + 24(4i+1); \tag{3.1}$$

for even integers the rows of the cubes are given by the third order homogeneous recurrence relation

$$R_{i+2} = 3(R_{i+1} - R_i) + R_{i-1} + 12; \tag{3.2}$$

Even cubes $\in \bar{0}_4$ as $\bar{2}_4$ has no powers.

For $\bar{3}_4$ the rows of the cubes are given by the second order homogeneous recurrence relation

$$R_{i+1} = 2R_i - R_{i-1} + 24(4i+3); \tag{3.3}$$

Unlike the case with the squares, the rows of the odd and even cubes are incompatible. Hence the equation (3.4) is structurally impossible:

$$c^3 = a^3 + b^3 \tag{3.4}$$

For example, the possible class structures are

$$\left.\begin{array}{ll} \bar{1}_4 = \bar{0}_4 + \bar{1}_4 & (i) \\ \bar{3}_4 = \bar{0}_4 + \bar{3}_4 & (ii) \end{array}\right\} \tag{3.5}$$

or, in terms of the rows:

$$\left.\begin{array}{ll} R_1 = R_0 + R_1^{/} & (i) \\ R_3 = R_0 + R_3^{/} & (ii) \end{array}\right\}\Bigg\} \tag{3.6}$$

The patterns for the recurrence relations in (3.6) are

$$2R_i - R_{i-1} + 24\times 4i = 2R_k - R_{k-1} + 24\times 4k + 3(R_{j+1} - R_j) + R_{j-1} + 12. \tag{3.7}$$

Obviously, the row structures on the left hand sides are incompatible with those on the right hand sides. This shows that, unlike the squares, the rows of an even plus odd cube yield a row structure that does not match that of a cube.

4 Final comments

The elements of the rows of higher odd powers are members of Pellian-type sequences which, again, are incompatible for even and odd integers. For example, odd fifth powers have the row structure of the recurrence relation:

$$R_{i+1} = 5R_i - 7R_{i-1} + 3R_{i-2} - 480(4r_i + 1),\ Class\,\bar{1}_4; \tag{4.1}$$

However, elements in even rows do not seem to belong to recursive sequences which means that the rows are incompatible. The same applies to even powers with an odd factor in the power, such as $(c^2)^3 = c^6$.

Since elements of the rows of odd squares are also related to the triangular numbers T_n or pentagonal numbers D_n according to whether 3 does or does not divide N, respectively, it is not surprising that the triangular and pentagonal numbers also satisfy Pellian non-homogeneous recurrence relations:

$$T_{n+1} = 2T_n - T_{n-1} + 1 \tag{4.2}$$

$$D_{n+1} = 2D_n - D_{n-1} + 3 \tag{4.3}$$

respectively, as particular cases of Equation (2.2).

Powers of 2^n ($n > 1$) have been discussed previously [7] and a similar incompatibility exists; for triples of this type, the REDs are the important structural constraints [7]. Thus, ISA can simplify apparently complex systems by showing why certain equations are invalid. It is a natural artefact of the multifarious nature of the structures themselves.

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