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Isomorphism testings for graph $C_G(a,b)$

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Abstract: Let G be a finite abelian group such that $G \neq \{0\}$ and $a, b \in G \setminus \{0\}$. Let $C_G(a,b)$ be the graph whose vertex set is G and the edge set is given by

$$E = \{\{x, x+a\}, \{x, x+b\}, \{x, x-a\}, \{x, x-b\} : x \in G\}.$$

In this work, we use the properties of finite abelian group to derive isomorphism testing on the graph $C_G(a,b)$ defined above. We study classes of isomorphic graphs. This work generalizes Nicoloso and Pietropaoli's paper [2].
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1 Introduction

Let G be a finite abelian group such that $G \neq \{0\}$ and $a, b \in G \setminus \{0\}$ with $o(b) \leq o(a)$. Let $C_G(a,b)$ be the undirected graph whose vertex set is G and the edge set is given by

$$E = \{\{x, x+a\}, \{x, x+b\}, \{x, x-a\}, \{x, x-b\} : x \in G\}.$$

We shall assume that $a \neq \pm b$, otherwise $C_G(a,b)$ degenerates into $C_G(a) = C_G(b)$, where $C_G(a)$ is the graph whose vertex set is G and the edge set is given by $E = \{\{x, x+a\}, \{x, x-a\} : x \in G\}$. A connected component of the graph $C_G(a)$ is called an *a-cycle*.

A graph is *k-regular* if all its vertices have the same degree k. Under the above conditions, we can classify $C_G(a,b)$ into three types of regular graph as follows.

Theorem 1.1. *Let $a, b \in G \setminus \{0\}$ with $a \neq \pm b$.*

(1) $C_G(a,b)$ is 2-regular if and only if a and b are elements of order two.

(2) $C_G(a,b)$ is 3-regular if and only if either a or b (not both) is an element of order two.

(3) $C_G(a,b)$ is 4-regular if and only if a and b are not elements of order two.

Remark. When $G = \langle g \rangle$ is cyclic, G has a unique element of order 2, so $C_G(a,b)$ is not 2-regular. Then it is either 3-regular or 4-regular.

Next, we give a condition for $C_G(a,b)$ to be connected.

Theorem 1.2. *The graph $C_G(a,b)$ is connected if and only if the group G is generated by a and b.*

Proof. Assume that $C_G(a,b)$ is connected. Let $y \in G$ and $y \neq 0$. Then there is a path between vertices 0 and y, so $y = ka + lb$ for some $k, l \in \mathbb{Z}$. Thus, $y \in \langle a, b \rangle$. Hence, $G = \langle a, b \rangle$. Conversely, suppose that $G = \langle a, b \rangle$. Let x and y be two distinct vertices in $C_G(a,b)$. Then $y - x \in G = \langle a, b \rangle$. Thus, $y - x = ka + lb$ for some $k, l \in \mathbb{Z}$. This means that there is a path between x and y. Hence, $C_G(a,b)$ is connected. $\square$

Corollary 1.3. *Let $G = \langle g \rangle$ be a cyclic group. The graph $C_G(a,b)$ is connected if and only if $g = ka + lb$ for some $k, l \in \mathbb{Z}$.*

Proof. It directly follows from Theorem 1.2 because $G = \langle g \rangle = \langle a, b \rangle$ is equivalent to $g = ka + lb$ for some $k, l \in \mathbb{Z}$. $\square$

Two graphs (V, E) and (V', E') are said to be *isomorphic*, denoted by $(V, E) \simeq (V, E)$, if there exists a bijection $f : V \to V'$ such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$ for all $x, y \in V$. Note that $C_G(a,b)$ and $C_G(b,a)$ are trivially isomorphic. Moreover, since the edge sets of the graphs $C_G(a,b)$, $C_G(-a,b)$, $C_G(a,-b)$, $C_G(-a,-b)$ are the same set, they are also isomorphic.

We can generalize Theorem 1.2 as follows.

Theorem 1.4. *If $H = \langle a, b \rangle$, then the graph $C_G(a,b)$ has $[G : H] = |G|/|H|$ connected components, each of which is isomorphic to $C_H(a,b)$.*

Proof. Let $x \in G$ and let $x + C_H(a,b)$ be the translation graph whose vertex set is $x + H$ and edge set is $\{\{x+h, x+h+a\}, \{x+h, x+h+b\}, \{x+h, x+h-a\}, \{x+h, x+h-b\} : h \in H\}$. Clearly, $x + C_H(a,b)$ is isomorphic to $C_H(a,b)$. By Theorem 1.2, $C_H(a,b)$ is connected, so $x + C_H(a,b)$ is a connected component of $C_G(a,b)$ for all $x \in G$. Since $\bigcup_{x \in G} \big(x + C_H(a,b)\big) = C_G(a,b)$ and $|\{x + C_H(a,b) : x \in G\}| = |\{x + H : x \in G\}| = [G : H] = |G|/|H|$, we have $C_G(a,b)$ has $|G|/|H|$ connected components and each component is isomorphic to $C_H(a,b)$. $\square$

Remark. If $G = \mathbb{Z}_n$ is a cyclic group of order $n \geq 2$, then $H = \langle a, b \rangle = \langle \gcd(a,b) \rangle$, so $|H| = \frac{n}{\gcd(n, \gcd(a,b))} = \frac{n}{\gcd(n,a,b)}$ and $C_G(a,b)$ has $\gcd(n,a,b)$ connected components.

Furthermore, for a cyclic group G, Nicoloso and Pietropaoli [2] studied the isomorphism testing problem for connected circulant graphs $C_G(a,b)$ and derived a necessary and sufficient condition to test whether two circulant graphs $C_G(a,b)$ and $C_G(a',b')$ are isomorphic. They proposed an elementary method to solve isomorphism testing, which is purely combinatorial and new for the problem. In addition, properties of the classes of mutually isomorphic graphs were analyzed.

In this work, we let G be any finite abelian group and use their properties to define the representative matrix of $C_H(a,b)$ and derive isomorphism testing on the graph $C_G(a,b)$ defined above. We study classes of isomorphic graphs. This generalizes Nicoloso and Pietropaoli's paper [2].

The paper is organized as follows. In the next section, we represent our graph $C_G(a,b)$ as the matrix $M(a,b)$ and study its properties including a-cycles, b-cycles, column jumps and block jumps. Isomorphism criteria and examples are given in the final section.

2 Cycles and matrices

In the previous section, we learn that each connected components of $C_G(a,b)$ is isomorphic to $C_H(a,b)$ where $H = \langle a,b\rangle$. Now we start with the definition of representative matrix of $C_H(a,b)$ denote by $M(a,b)$, which will be used to prove the isomorphism testing as our main theorem in the next section. The representative matrix $M(a,b)$ for the graph $C_H(a,b)$ can be defined as the following table.

0	a	$2a$	$\dots$	$(o(a)-1)a$
b	$b+a$	$b+2a$	$\dots$	$b+(o(a)-1)a$
$2b$	$2b+a$	$2b+2a$	$\dots$	$2b+(o(a)-1)a$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(o(b+\langle a\rangle)-1)b$	$(o(b+\langle a\rangle)-1)b+a$	$(o(b+\langle a\rangle)-1)b+2a$	$\dots$	$(o(b+\langle a\rangle)-1)b+(o(a)-1)a$

Lemma 2.1. *Let $a,b \in G \smallsetminus \{0\}$ with $a \neq \pm b$ and $H = \langle a,b\rangle$. Then*

$$H/\langle a\rangle = \{\langle a\rangle, b+\langle a\rangle, 2b+\langle a\rangle, \dots, (o(b+\langle a\rangle)-1)b+\langle a\rangle\} = \langle b+\langle a\rangle\rangle.$$

In particular, $o(b+\langle a\rangle) = \frac{|H|}{o(a)}$.

Proof. Clearly, $\langle b+\langle a\rangle\rangle \subset H/\langle a\rangle$. Let $x \in H$. Then $x = ka+lb$ for some $k,l \in \mathbb{Z}$, so $x+\langle a\rangle = ka+lb+\langle a\rangle = lb+\langle a\rangle \in \langle b+\langle a\rangle\rangle$. Hence, $H/\langle a\rangle = \langle b+\langle a\rangle\rangle$. Moreover, $o(b+\langle a\rangle) = |\langle b+\langle a\rangle\rangle| = |H/\langle a\rangle| = \frac{|H|}{o(a)}$. □

From the above matrix, $M(a,b)$ has $r = o(b+\langle a\rangle)$ rows and $c = o(a)$ columns. The number of entries of $M(a,b)$ is $o(b+\langle a\rangle)o(a) = |H|$. Each row corresponds to a coset in the quotient $H/\langle a\rangle$ and all entries of $M(a,b)$ are distinct. In other words, vertices of $C_H(a,b)$ appear exactly once.

Two vertices $x,y \in G$ are said to be *a-adjacent* if $y-x = \pm a$ and $\{x,y\}$ is said to be an *a-edge* and x,y are in an a-cycle if $y-x \in \langle a\rangle$. Notice that two consecutive entries of a row are a-adjacent and the first and the last entries of a same row also are a-adjacent, so that each row of $M(a,b)$ corresponds to an a-cycle of $C_H(a,b)$. Thus, $C_H(a,b)$ consists of $o(b+\langle a\rangle)$ a-cycles of length $o(a)$. In addition, two consecutive entries of a column are b-adjacent, that is, their difference is $\pm b$. However, the first and the last entries of a same column are not necessarily b-adjacent. It depends on the *column-jump* of $M(a,b)$ denoted by $\lambda(a,b)$.

Lemma 2.2. *Let $a, b \in G \setminus \{0\}$. Then there exists a unique number $\lambda(a,b) \in \{0, 1, \ldots, o(a)-1\}$, called the column-jump of $M(a,b)$, satisfying*

$$rb = \lambda(a,b)a. \tag{1}$$

Proof. Since $b + \langle a \rangle \in G/\langle a \rangle$, we have

$$rb + \langle a \rangle = o(b + \langle a \rangle)b + \langle a \rangle = o(b + \langle a \rangle)(b + \langle a \rangle) = \langle a \rangle,$$

so $rb \in \langle a \rangle$. Hence, there exists a unique $\lambda(a,b) \in \{0, 1, \ldots, o(a) - 1\}$ such that $rb = \lambda(a,b)a$ as desired. □

Some remarks on the column-jump of $M(a,b)$ are studied in the next theorem.

Theorem 2.3. *Let $a, b \in G \setminus \{0\}$.*

(1) $\lambda(-a,b) = 0$ if and only if $\lambda(a,b) = 0$.

(2) $\lambda(-a,b) = o(a) - \lambda(a,b)$ if $\lambda(a,b)$ and $\lambda(-a,b)$ are nonzero.

(3) $\lambda(a,-b) = 0$ if and only if $\lambda(a,b) = 0$.

(4) $\lambda(a,-b) = o(a) - \lambda(a,b)$ if $\lambda(a,b)$ and $\lambda(a,-b)$ are nonzero.

(5) $\lambda(-a,-b) = \lambda(a,b)$.

Theorem 2.4. *Let $a, b \in G$. Then $\dfrac{c}{o(a + \langle b \rangle)} = \dfrac{o(b)}{r}$.*

Proof. If a or $b = 0$, the conclusion is trivial. Assume that $a, b \in G \setminus \{0\}$. From Lemma 2.1, we have $r = \dfrac{|H|}{c}$ and $o(a+\langle b \rangle) = \dfrac{|H|}{o(b)}$. Then $rc = |H| = o(a+\langle b \rangle)o(b)$, so $\dfrac{c}{o(a + \langle b \rangle)} = \dfrac{o(b)}{r}$. □

Theorem 2.5. *Let $a, b \in G \setminus \{0\}$ and write $\lambda = \lambda(a,b) \neq 0$. Then $\gcd(\lambda, o(a)) = o(a + \langle b \rangle)$.*

Proof. From Eq. (1) and $r = o(b + \langle a \rangle) \mid o(b)$,

$$\frac{o(b)}{r} = \frac{o(b)}{\gcd(r, o(b))} = o(rb) = o(\lambda a) = \frac{o(a)}{\gcd(\lambda, o(a))}.$$

Thus, we have $\gcd(\lambda, o(a)) = o(a) \cdot \frac{r}{o(b)} = o(a + \langle b \rangle)$ by Theorem 2.4. □

From the above theorem, $\langle \lambda a \rangle = \{0, \lambda a, 2\lambda a, \ldots, \big(\frac{o(a)}{o(a+\langle b \rangle)} - 1\big)\lambda a\}$. This implies that a b-cycle of $C_H(a,b)$ consists of $h = \frac{o(a)}{o(a+\langle b \rangle)} = \frac{c}{o(a+\langle b \rangle)}$ columns. As a consequence, $M(a,b)$ can be partitioned into h equally sized submatrices, the blocks denoted by β_l where $l \in \{0, 1, \ldots, h-1\}$. The block β_l is defined on all the r rows and $o(a + \langle b \rangle)$ consecutive columns from column $lo(a + \langle b \rangle) + 1$ to $(l+1)o(a + \langle b \rangle)$.

Since $o(a + \langle b \rangle) \mid \lambda(a,b)$, that is, $\lambda(a,b)$ is a multiple of $o(a + \langle b \rangle)$. From this, we define the *block-jump* of $M(a,b)$ to be

$$\Lambda(a,b) = \frac{\lambda(a,b)}{o(a + \langle b \rangle)}, \tag{2}$$

where $\Lambda(a,b) \in \{0, 1, \ldots, h-1\}$. Moreover, we have:

Theorem 2.6. *Let $a, b \in G \setminus \{0\}$.*

(1) $\Lambda(-a, b) = h - \Lambda(a, b)$.

(2) $\Lambda(a, -b) = h - \Lambda(a, b)$.

(3) $\Lambda(-a, -b) = \Lambda(a, b)$.

Proof. We use the definition of block-jump and Theorem 2.3 to prove (1)–(3) as follows.

(1) $\Lambda(-a, b) = \frac{\lambda(-a,b)}{o(-a+\langle b\rangle)} = \frac{o(a)-\lambda(a,b)}{o(a+\langle b\rangle)} = \frac{o(a)}{o(a+\langle b\rangle)} - \frac{\lambda(a,b)}{o(a+\langle b\rangle)} = h - \Lambda(a, b)$.

(2) $\Lambda(a, -b) = \frac{\lambda(a,-b)}{o(a+\langle -b\rangle)} = \frac{o(a)-\lambda(a,b)}{o(a+\langle b\rangle)} = \frac{o(a)}{o(a+\langle b\rangle)} - \frac{\lambda(a,b)}{o(a+\langle b\rangle)} = h - \Lambda(a, b)$.

(3) $\Lambda(-a, -b) = \frac{\lambda(-a,-b)}{o(-a+\langle -b\rangle)} = \frac{\lambda(a,b)}{o(a+\langle b\rangle)} = \Lambda(a, b)$.

This completes the proof. □

Example 2.7. Since $\mathbb{Z}_2 \times \mathbb{Z}_6 = \langle (1, 2), (0, 3)\rangle$, the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1, 2), (0, 3))$ is connected. We have $r = o((0, 3) + \langle (1, 2)\rangle) = 2$ and

$$(0, 0) = 2(0, 3) = r(0, 3) = \lambda((1, 2), (0, 3))(1, 2),$$

so $\lambda((1, 2), (0, 3)) = 0$, which implies $\Lambda((1, 2), (0, 3)) = \frac{\lambda((1,2),(0,3))}{o((1,2)+\langle (0,3)\rangle)} = \frac{0}{6} = 0$. Since $c = o((1, 2)) = 6$, $M((1, 2), (0, 3))$ has $r = 2$ rows and $c = 6$ columns. The representative matrix $M((1, 2), (0, 3))$ for the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1, 2), (0, 3))$ is the following table (the blocks are separated by double lines)

$(0, 0)$	$(1, 2)$	$(0, 4)$	$(1, 0)$	$(0, 2)$	$(1, 4)$
$(0, 3)$	$(1, 5)$	$(0, 1)$	$(1, 3)$	$(0, 5)$	$(1, 1)$

Example 2.8. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (0, 1, 2), (1, 1, 1)\rangle$, the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0, 1, 2), (1, 1, 1))$ is connected. We have $r = o((1, 1, 1) + \langle (0, 1, 2)\rangle) = 2$ and

$$(0, 0, 2) = 2(1, 1, 1) = r(1, 1, 1) = \lambda((0, 1, 2), (1, 1, 1))(0, 1, 2),$$

so $\lambda((0, 1, 2), (1, 1, 1)) = 4$, which implies $\Lambda((0, 1, 2), (1, 1, 1)) = \frac{\lambda((0,1,2),(1,1,1))}{o((0,1,2)+\langle (1,1,1)\rangle)} = \frac{4}{2} = 2$. Since $c = o((0, 1, 2)) = 6$, $M((0, 1, 2), (1, 1, 1))$ has $r = 2$ rows and $c = 6$ columns. The representative matrix $M((0, 1, 2), (1, 1, 1))$ for the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0, 1, 2), (1, 1, 1))$ is the following table (the blocks are separated by double lines)

$(0, 0, 0)$	$(0, 1, 2)$	$(0, 0, 1)$	$(0, 1, 0)$	$(0, 0, 2)$	$(0, 1, 1)$
$(1, 1, 1)$	$(1, 0, 0)$	$(1, 1, 2)$	$(1, 0, 1)$	$(1, 1, 0)$	$(1, 0, 2)$

3 Isomorphism Theorem

In the previous section, we define the representative matrix $M(a,b)$ for the graph $C_H(a,b)$, which has $r = o(b+\langle a\rangle)$ rows and $c = o(a)$ columns. Moreover, we define the block-jump of $M(a,b)$ denoted by $\Lambda(a,b)$, which is a constant in $\{0,\ldots,h-1\}$, where $h = \frac{o(a)}{o(a+\langle b\rangle)}$. In this section, we study the isomorphism testing problem for the graphs $C_H(a,b)$ and use the properties of $M(a,b)$ to derive a necessary and sufficient condition to test whether two graphs $C_G(a,b)$ and $C_G(a',b')$ are isomorphic. Our main theorem is as follows.

Theorem 3.1. *Let $a,a',b,b' \in G \setminus \{0\}$. Then $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic if and only if either one of the following two conditions holds:*

1. *$r = r'$, $o(b') = o(b) < c = c'$ and $\Lambda(a,b) = \pm\Lambda(a',b')$;*

2. *$r = r'$, $o(b') = o(b) = c = c'$ and either $\Lambda(a,b) = \pm\Lambda(a',b')$ or $\Lambda(a,b) = \pm\Lambda(b',a')$,*

where $H = \langle a,b\rangle, H' = \langle a',b'\rangle, r = o(b+\langle a\rangle), r' = o(b'+\langle a'\rangle), c = o(a), c' = o(a'), \Lambda(a,b) = \frac{\lambda(a,b)}{o(a+\langle b\rangle)}$ and $\Lambda(a',b') = \frac{\lambda(a',b')}{o(a'+\langle b'\rangle)}$, where $\lambda(a,b)$ and $\lambda(a',b')$ are the column-jump of $M(a,b)$ and $M(a',b')$ respectively.

Proof. *Case 1.* *$r = r'$, $o(b') = o(b) < c = c'$ and $\Lambda(a,b) = \pm\Lambda(a',b')$.* By Theorem 2.4, $o(a+\langle b\rangle) = \frac{r}{o(b)}\cdot c = \frac{r'}{o(b')}\cdot c' = o(a'+\langle b'\rangle)$ and observe that $h = \frac{c}{o(a+\langle b\rangle)} = \frac{c'}{o(a'+\langle b'\rangle)} = h'$. Then $M(a,b)$ and $M(a',b')$ have the same number of rows and columns and the same size of the blocks.

1.1 $\Lambda(a,b) = \Lambda(a',b')$. From the representative matrices $M(a,b)$ and $M(a',b')$, we see that a-edge $\{ib+ja, ib+(j+1)a\}$ is mapped onto the homologous a'-edge $\{ib'+ja', ib'+(j+1)a'\}$ for all $i \in \{0,1,\ldots,r-1\}$ and $j \in \{0,1,\ldots,c-1\}$. A b-edge $\{ib+ja,(i+1)b+ja\}$ is mapped onto the homologous b'-edge $\{ib'+ja',(i+1)b'+ja'\}$ for all $i \in \{0,1,\ldots,r-2\}$ and $j \in \{0,1,\ldots,c-1\}$. While, the another b-edge connecting an entry of the last row with an entry of the first row of $M(a,b)$, namely $\{(r-1)b+ja,(j+\Lambda(a,b)o(a+\langle b\rangle))a\}$ is mapped onto the homologous b'-edge $\{(r-1)b'+ja',(j+\Lambda(a',b')o(a'+\langle b'\rangle))a'\}$ for all $j \in \{0,1,\ldots,c-1\}$. For these reasons, there exists a bijection from $ib+ja$ maps onto the homologous $ib'+ja'$ such that the adjacencies are preserved, for all $i \in \{0,1,\ldots,r-1\}$ and $j \in \{0,1,\ldots,c-1\}$. Hence, $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic.

1.2 $\Lambda(a,b) = -\Lambda(a',b')$. Consider the graph $C_{H'}(-a',b')$, its block-jump is $\Lambda(-a',b') = h-\Lambda(a',b') = \Lambda(a,b)$. By applying the previous case, since $\Lambda(-a',b') = \Lambda(a,b)$, $C_{H'}(-a',b')$ and $C_H(a,b)$ are isomorphic. Since $C_{H'}(-a',b')$ and $C_{H'}(a',b')$ are trivially isomorphic, $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic.

Case 2. *$r = r'$, $o(b') = o(b) = c = c'$ and either $\Lambda(a,b) = \pm\Lambda(a',b')$ or $\Lambda(a,b) = \pm\Lambda(b',a')$.* Clearly, if $\Lambda(a,b) = \pm\Lambda(a',b')$, then $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Suppose $\Lambda(a,b) = \pm\Lambda(b',a')$. Then we can apply Case 1 by swapping a' and b', so $C_H(a,b)$

and $C_{H'}(b',a')$ are isomorphic. Since $C_{H'}(b',a')$ and $C_{H'}(a',b')$ are trivially isomorphic, $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic, as desired.

Conversely, assume that $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Then $|H| = |H'|$ and there exists a bijection f from H to H' as shown in the following tables.

	$\vdots$	$\vdots$	$\vdots$	
$\ldots$	$(i-1)b+(j-1)a$	$(i-1)b+ja$	$(i-1)b+(j+1)a$	$\ldots$
$\ldots$	$ib+(j-1)a$	$ib+ja$	$ib+(j+1)a$	$\ldots$
$\ldots$	$(i+1)b+(j-1)a$	$(i+1)b+ja$	$(i+1)b+(j+1)a$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	

	$\vdots$	$\vdots$	$\vdots$	
$\ldots$	$f((i-1)b+(j-1)a)$	$f((i-1)b+ja)$	$f((i-1)b+(j+1)a)$	$\ldots$
$\ldots$	$f(ib+(j-1)a)$	$f(ib+ja)$	$f(ib+(j+1)a)$	$\ldots$
$\ldots$	$f((i+1)b+(j-1)a)$	$f((i+1)b+ja)$	$f((i+1)b+(j+1)a)$	$\ldots$
	$\vdots$	$\vdots$	$\vdots$	

for all $i, j \in \mathbb{Z}$. Let E' be the edge set of $C_{H'}(a',b')$. We shall consider the entries in the second table. Assume that $f(ib+ja) = 0$ for some $i \in \{0, 1, \ldots, r-1\}$ and $j \in \{0, 1, \ldots, c-1\}$.

	$f((i-1)b+ja)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$
	$f((i+1)b+ja)$	

Since $\{0, a'\} \in E'$, we may assume that $a' = f(ib+(j+1)a)$.

	$f((i-1)b+ja)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$
	$f((i+1)b+ja)$	

Since $\{0, b'\} \in E'$, either $b' = f((i+1)b+ja)$ or $b' = f((i-1)b+ja)$ or $b' = f(ib+(j-1)a)$. Suppose $b' = f(ib+(j-1)a)$.

	$f((i-1)b+ja)$	
$f(ib+(j-1)a)$ $=b'$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$
	$f((i+1)b+ja)$	

Since $\{b', b'+a'\}, \{a', b'+a'\} \in E'$, $b'+a' = 0$, which contradicts $a' \neq \pm b'$. Thus, we may assume that $b' = f((i+1)b+ja)$. Clearly, $b'+a' = f((i+1)b+(j+1)a)$.

	$f((i-1)b+ja)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$
	$f((i+1)b+ja)$ $=b'$	$f((i+1)b+(j+1)a)$ $=b'+a'$

Since $\{a', 2a'\} \in E'$, either $2a' = f(ib+(j+2)a)$ or $2a' = f((i-1)b+(j+1)a)$. Suppose $2a' = f((i-1)b+(j+1)a)$.

	$f((i-1)b+ja)$	$f((i-1)b+(j+1)a)$ $=2a'$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$	$f(ib+(j+2)a)$
	$f((i+1)b+ja)$ $=b'$	$f((i+1)b+(j+1)a)$ $=b'+a'$	

Since $\{2a', b'+2a'\}, \{b'+a', b'+2a'\} \in E'$, $b'+2a' = a'$, which contradicts $a' \neq \pm b'$. So $2a' = f(ib+(j+2)a)$. Clearly, $b'+2a' = f((i+1)b+(j+2)a)$.

	$f((i-1)b+ja)$	$f((i-1)b+(j+1)a)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$	$f(ib+(j+2)a)$ $=2a'$
	$f((i+1)b+ja)$ $=b'$	$f((i+1)b+(j+1)a)$ $=b'+a'$	$f((i+1)b+(j+2)a)$ $=b'+2a'$

Since $\{b', 2b'\} \in E'$, either $2b' = f((i+2)b+ja)$ or $2b' = f((i+1)b+(j-1)a)$. Suppose $2b' = f((i+1)b+(j-1)a)$.

	$f((i-1)b+ja)$	$f((i-1)b+(j+1)a)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$	$f(ib+(j+2)a)$ $=2a'$
$f((i+1)b+(j-1)a)$ $=2b'$	$f((i+1)b+ja)$ $=b'$	$f((i+1)b+(j+1)a)$ $=b'+a'$	$f((i+1)b+(j+2)a)$ $=b'+2a'$
	$f((i+2)b+ja)$		

Since $\{2b', 2b'+a'\}, \{b'+a', 2b'+a'\} \in E'$, $2b'+a' = b'$, which contradicts $a' \neq \pm b'$. So $2b' = f((i+2)b+ja)$, which implies $2b'+a' = f((i+2)b+(j+1)a)$ and $2b'+2a' = f((i+2)b+(j+2)a)$. Hence, we have the following nine entries.

	$f((i-1)b+ja)$	$f((i-1)b+(j+1)a)$	
$f(ib+(j-1)a)$	$f(ib+ja)$ $=0$	$f(ib+(j+1)a)$ $=a'$	$f(ib+(j+2)a)$ $=2a'$
$f((i+1)b+(j-1)a)$	$f((i+1)b+ja)$ $=b'$	$f((i+1)b+(j+1)a)$ $=b'+a'$	$f((i+1)b+(j+2)a)$ $=b'+2a'$
	$f((i+2)b+ja)$ $=2b'$	$f((i+2)b+(j+1)a)$ $=2b'+a'$	$f((i+2)b+(j+2)a)$ $=2b'+2a'$

We shall show that

$$f(mb + na) = (m - i)b' + (n - j)a'$$

for all $m \in \{i, i+1, \ldots, i+r-1\}$ and $n \in \{j, j+1, \ldots, j+c-1\}$ by the mathematical induction. The basic step of the statement is obtained from above table. Let $k \in \{i, i+1, \ldots, i+r-2\}$ and $l \in \{j, j+1, \ldots, j+c-2\}$. Assume that $f(ub + va) = (u - i)b' + (v - j)a'$ for all $u \in \{i, i+1, \ldots, k\}$ and $v \in \{j, j+1, \ldots, l\}$. The induction hypotheses say that

$$\begin{aligned}
f(kb + la) &= (k - i)b' + (l - j)a' \\
&= (k - i - 1)b' + (l - j)a' + b' \\
&= f((k - 1)b + la) + b', \\
f((k - 2)b + la) &= (k - i - 2)b' + (l - j)a' \\
&= (k - i - 1)b' + (l - j)a' - b' \\
&= f((k - 1)b + la) - b' \text{ and} \\
f((k - 1)b + (l - 1)a) &= (k - i - 1)b' + (l - j - 1)a' \\
&= (k - i - 1)b' + (l - j)a' - a' \\
&= f((k - 1)b + la) - a'.
\end{aligned}$$

That is,

	$f((k-2)b+la)$ $= f((k-1)b+la)-b'$	
$f((k-1)b+(l-1)a)$ $= f((k-1)b+la)-a'$	$f((k-1)b+la)$	$f((k-1)b+(l+1)a)$
	$f(kb+la)$ $= f((k-1)b+la)+b'$	

Then $f((k-1)b + (l+1)a) = f((k-1)b + la) + a'$.

	$f((k-2)b+la)$ $= f((k-1)b+la)-b'$	
$f((k-1)b+(l-1)a)$ $= f((k-1)b+la)-a'$	$f((k-1)b+la)$	$f((k-1)b+(l+1)a)$ $= f((k-1)b+la)+a'$
	$f(kb+la)$ $= f((k-1)b+la)+b'$	

From

$$\begin{aligned}
f(kb + (l - 1)a) &= (k - i)b' + (l - j - 1)a' \\
&= (k - i)b' + (l - j)a' - a' \\
&= f(kb + la) - a' \text{ and} \\
f((k - 1)b + la) &= (k - i - 1)b' + (l - j)a' \\
&= (k - i)b' + (l - j)a' - b' \\
&= f(kb + la) - b',
\end{aligned}$$

so either $f(kb + (l+1)a) = f(kb + la) + a'$ or $f(kb + (l+1)a) = f(kb + la) + b'$, we suppose that $f(kb + (l+1)a) = f(kb + la) + b'$.

	$f((k-2)b+la)$	
$f((k-1)b+(l-1)a)$	$f((k-1)b+la)$ $= f(kb+la)-b'$	$f((k-1)b+(l+1)a)$ $= f((k-1)b+la)+a'$
$f(kb+(l-1)a)$ $= f(kb+la)-a'$	$f(kb+la)$	$f(kb+(l+1)a)$ $= f(kb+la)+b'$
	$f((k+1)b+la)$	

Note that

$$\begin{aligned} f(kb+la)+b' &= (k-i+1)b' + (l-j)a' \\ &= (k-i-1)b' + (l-j+1)a' - a' + 2b' \\ &= f((k-1)b+la) + a' - a' + 2b', \end{aligned}$$

so $\{f(kb+la)+b', f((k-1)b+la)+a'\} \notin E'$, which contradicts the above table. So $f(kb+(l+1)a) = f(kb+la)+a'$. Thus, $f((k+1)b+la) = f(kb+la)+b'$.

	$f((k-2)b+la)$	
$f((k-1)b+(l-1)a)$	$f((k-1)b+la)$ $= f(kb+la)-b'$	$f((k-1)b+(l+1)a)$ $= f((k-1)b+la)+a'$
$f(kb+(l-1)a)$ $= f(kb+la)-a'$	$f(kb+la)$	$f(kb+(l+1)a)$ $= f(kb+la)+a'$
	$f((k+1)b+la)$ $= f(kb+la)+b'$	$f((k+1)b+(l+1)a)$

Hence $f((k+1)b+(l+1)a) = f(kb+la)+a'+b' = (k-i+1)b' + (l-j+1)a'$, as claimed.

Next, we prove that $c = c'$. From

$$ca' = f(ib+(j+c)a) = f(ib+ja) = 0 = c'a',$$

so $(c-c')a' = 0$. Then $c' = o(a')$ divides $c - c'$, so $c' \mid c$. On the other hand, since f is injective and

$$f(ib+ja) = 0 = c'a' = f(ib+(j+c')a),$$

$ib+ja = ib+(j+c')a$. Thus, $c'a = 0$, which implies $c \mid c'$. Hence, we conclude that $c = c'$. Since $rc = |H| = |H'| = r'c'$, we have $r = r'$. Note that

$$\begin{aligned} \lambda(a',b')a' = r'b' = rb' &= f((i+r)b+ja) \\ &= f((i+r)b+(\lambda(a,b)-\lambda(a,b)+j)a) \\ &= f(ib+(j+\lambda(a,b))a) = \lambda(a,b)a', \end{aligned}$$

where the last line of above follows from Eq. (1), so $\lambda(a,b) = \lambda(a',b')$. By Theorem 2.5, since $o(a) = c = c' = o(a')$ and $\lambda(a,b) = \lambda(a',b')$,

$$o(a+\langle b\rangle) = \gcd(\lambda(a,b), o(a)) = \gcd(\lambda(a',b'), o(a')) = o(a'+\langle b'\rangle),$$

which implies $o(b) = \frac{rc}{o(a+\langle b\rangle)} = \frac{r'c'}{o(a'+\langle b'\rangle)} = o(b')$ by Theorem 2.4. Recall that $o(b) \le o(a) = c$, so we can distinguish the following two cases.

Case 1. $o(b) = c$.

1.1 $a' = f(ib + (j+1)a)$ *and* $b' = f((i+1)b + ja)$.

	$f((i-1)b + ja)$	
$f(ib + (j-1)a)$	$f(ib + ja)$ $= 0$	$f(ib + (j+1)a)$ $= a'$
	$f((i+1)b + ja)$ $= b'$	

From the above proof, we have $r = r'$ and $o(b') = o(b) = c = c'$. Eq. (2) gives

$$\Lambda(a,b) = \frac{\lambda(a,b)}{o(a + \langle b \rangle)} = \frac{\lambda(a',b')}{o(a' + \langle b' \rangle)} = \Lambda(a',b').$$

1.2 $a' = f(ib + (j+1)a)$ *and* $b' = f((i-1)b + ja)$.

	$f((i-1)b + ja)$ $= b'$	
$f(ib + (j-1)a)$	$f(ib + ja)$ $= 0$	$f(ib + (j+1)a)$ $= a'$
	$f((i+1)b + ja)$ $= -b'$	

Since $C_H(a,b)$ corresponds to $C_{H'}(a',b') \simeq C_{H'}(a',-b')$, there exists a bijection f from $H = \langle a, b \rangle$ to $H' = \langle a', -b' \rangle$. Thus, we can apply the previous case by substituting $-b'$ in b', we have $r = o(b + \langle a \rangle) = o(-b' + \langle a' \rangle) = o(b' + \langle a' \rangle) = r'$ and $o(b') = o(-b') = o(b) = c = c'$. From Theorem 2.6 (2), $\Lambda(a,b) = \Lambda(a',-b') = h - \Lambda(a',b') = -\Lambda(a',b')$.

1.3 $a' = f(ib + (j-1)a)$ *and* $b' = f((i+1)b + ja)$.

1.4 $a' = f(ib + (j-1)a)$ *and* $b' = f((i-1)b + ja)$.

We can prove Cases 1.3 and 1.4 similar to Case 1.2. So $r = r'$, $o(b') = o(b) = c = c'$ and $\Lambda(a,b) = \pm\Lambda(a',b')$.

1.5 $a' = f((i+1)b + ja)$ *and* $b' = f(ib + (j+1)a)$.

	$f((i-1)b + ja)$	
$f(ib + (j-1)a)$	$f(ib + ja)$ $= 0$	$f(ib + (j+1)a)$ $= b'$
	$f((i+1)b + ja)$ $= a'$	

Since $C_H(a,b)$ corresponds to $C_{H'}(a',b') \simeq C_{H'}(b',a')$, there exists a bijection f from $H = \langle a, b \rangle$ to $H' = \langle b', a' \rangle$. Thus, we can apply Case 1.1 by swapping a' and b', we have $o(b') = o(b) = c = c'$ and $\Lambda(a,b) = \Lambda(b',a')$. By Theorem 2.4,

$$r = o(b + \langle a \rangle) = o(a' + \langle b' \rangle) = \frac{o(a' + \langle b' \rangle)}{c'} \cdot o(b') = r'.$$

1.6 $a' = f((i-1)b + ja)$ *and* $b' = f(ib + (j+1)a)$.

1.7 $a' = f((i+1)b + ja)$ *and* $b' = f(ib + (j-1)a)$.

1.8 $a' = f((i-1)b + ja)$ *and* $b' = f(ib + (j-1)a)$.

For Cases 1.6–1.8, we can prove similar to 1.5 by applying Cases 1.2–1.4 respectively. So $r = r', o(b') = o(b) = c = c'$ and $\Lambda(a,b) = \pm\Lambda(b',a')$.

Hence $r = r', o(b') = o(b) = c = c'$ and either $\Lambda(a,b) = \pm\Lambda(a',b')$ or $\Lambda(a,b) = \pm\Lambda(b',a')$, as desired.

Case 2. $o(b) < c$. Then we have four cases same as 1.1–1.4. So $r = r', o(b') = o(b) < c = c'$ and $\Lambda(a,b) = \pm\Lambda(a',b')$, as desired.

This completes this main theorem. ☐

Lemma 3.2. *Let* $a, a', b, b' \in G \setminus \{0\}$.

(1) If $r = r', o(b) = o(b')$ *and* $c = c'$*, then* $rb' = \pm\lambda(a,b)a'$ *if and only if* $\Lambda(a,b) = \pm\Lambda(a',b')$.

(2) If $r = r'$ *and* $o(b') = o(b) = c = c'$*, then* $ra' = \pm\lambda(a,b)b'$ *if and only if* $\Lambda(a,b) = \pm\Lambda(b',a')$.

Proof. (1) Let $r = r', o(b) = o(b')$ and $c = c'$. By Theorem 2.4, we have $o(a+\langle b\rangle) = o(a'+\langle b'\rangle)$. Assume that $rb' = \pm\lambda(a,b)a'$. Eq. (1) and $r = r'$ give $\lambda(a,b)a' = \pm\lambda(a',b')a'$, which implies $\lambda(a,b) = \pm\lambda(a',b')$. Then $\Lambda(a,b) = \frac{\lambda(a,b)}{o(a+\langle b\rangle)} = \pm\frac{\lambda(a',b')}{o(a'+\langle b'\rangle)} = \pm\Lambda(a',b')$ by Eq. (2).

Conversely, assume that $\Lambda(a,b) = \pm\Lambda(a',b')$. From Eq. (2), we have $\lambda(a,b) = \Lambda(a,b)o(a+\langle b\rangle) = \pm\Lambda(a',b')o(a'+\langle b'\rangle) = \pm\lambda(a',b')$, so

$$rb' = r'b' = \lambda(a',b')a' = \pm\lambda(a,b)a'$$

as desired.

(2) We can apply (1) by swapping a' and b'.
Hence, we have the lemma. ☐

We may deduce the following corollary from the above lemma.

Corollary 3.3. *Let* $a, a', b, b' \in G \setminus \{0\}$*. Then* $C_G(a,b)$ *and* $C_G(a',b')$ *are isomorphic if and only if either one of the following two conditions holds:*

(1) $r = r'$, $o(b') = o(b) < c = c'$ *and* $rb' = \pm\lambda(a,b)a'$*;*

(2) $r = r'$, $o(b') = o(b) = c = c'$ *and either* $rb' = \pm\lambda(a,b)a'$ *or* $ra' = \pm\lambda(a,b)b'$.

Proof. Let $H = \langle a,b\rangle$ and $H' = \langle a',b'\rangle$. From Lemma 2.1, $|H| = rc = r'c' = |H'|$. By Theorem 1.4, we have $C_G(a,b)$ and $C_G(a',b')$ are isomorphic if and only if $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Hence, this corollary follows from Theorem 3.1 and Lemma 3.2. ☐

We give some examples to demonstrate the above corollary.

Example 3.4. Let $a = (1,0), a' = (0,1), b = (0,2), b' = (2,0)$ be in $\mathbb{Z}_4 \times \mathbb{Z}_4$. Then $o(b') = o((2,0)) = 2 = o((0,2)) = o(b) < c = o((1,0)) = 4 = o((0,1)) = c'$ and $r = o((0,2) + \langle(1,0)\rangle) = 2 = o((2,0) + \langle(0,1)\rangle) = r'$. From Lemma 2.2, since

$$(0,0) = 2(0,2) = rb = \lambda(a,b)a = \lambda(a,b)(1,0)$$

for some $\lambda(a,b) \in \{0,1,2,3 = c-1\}$, $\lambda(a,b) = 0$. Thus,

$$rb' = 2(2,0) = (0,0) = 0(0,1) = \lambda(a,b)a'.$$

By Corollary 3.3, $C_{\mathbb{Z}_4\times\mathbb{Z}_4}((1,0),(0,2))$ is isomorphic to $C_{\mathbb{Z}_4\times\mathbb{Z}_4}((0,1),(2,0))$.

Example 3.5. Let $a = (1,0), a' = (1,1), b = (0,1), b' = (2,0)$ be in $\mathbb{Z}_4 \times \mathbb{Z}_4$. Since $o(b') = o((2,0)) = 2 \neq 4 = o((0,1)) = o(b)$, $C_{\mathbb{Z}_4\times\mathbb{Z}_4}((1,0),(0,1))$ and $C_{\mathbb{Z}_4\times\mathbb{Z}_4}((1,1),(2,0))$ are not isomorphic by Corollary 3.3.

We quote two results on finite abelian groups as follows.

Theorem 3.6. *[1] Let G be a finite abelian group. Then there exist integers $n_1, \ldots, n_t > 1$ such that $n_1 \mid n_2, n_2 \mid n_3, \ldots, n_{t-1} \mid n_t$ and*

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t},$$

where these integers are uniquely defined by G.

Theorem 3.7. *[1] Let $G_1, G_2, \ldots, G_t$ be finite abelian groups and $(a_1, a_2, \ldots, a_t) \in \prod_{i=1}^{t} G_i$. Then*

$$o((a_1, a_2, \ldots, a_t)) = \operatorname{lcm}(o(a_1), o(a_2), \ldots, o(a_t)),$$

where $o(a_i)$ denotes order of a_i in G_i for all $i \in \{1, 2, \ldots, t\}$.

The next corollary gives an easier way to compute the order of elements.

Corollary 3.8. *Let $a = (a_1, a_2, \ldots, a_t), b = (b_1, b_2, \ldots, b_t) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$ where $n_1, \ldots, n_t > 1$ and $n_1 \mid n_2, n_2 \mid n_3, \ldots, n_{t-1} \mid n_t$. Then*

(1) $o(a) = \operatorname{lcm}\left(\frac{n_1}{\gcd(n_1,a_1)}, \frac{n_2}{\gcd(n_2,a_2)}, \ldots, \frac{n_t}{\gcd(n_t,a_t)}\right)$.

(2) $|H| = \frac{o(a)\cdot o(b)}{|\langle a\rangle \cap \langle b\rangle|}$.

(3) $o(b + \langle a\rangle) = \frac{|H|}{o(a)}$.

Example 3.9. Let $a = (1,2), a' = (0,1), b = (0,3), b' = (1,0)$ be in $\mathbb{Z}_2 \times \mathbb{Z}_6$. By Example 2.7, we have $r = o((0,3) + \langle(1,2)\rangle) = 2$ and $\lambda((1,2),(0,3)) = 0$, so

$$rb' = 2(1,0) = (0,0) = 0(0,1) = \lambda(a,b)a'.$$

From

$$\begin{aligned}
c = o(a) = o((1,2)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{6}{\gcd(6,2)}\right) = \operatorname{lcm}(2,3) = 6,\\
c' = o(a') = o((0,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{6}{\gcd(6,1)}\right) = \operatorname{lcm}(1,6) = 6,\\
o(b) = o((0,3)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{6}{\gcd(6,3)}\right) = \operatorname{lcm}(1,2) = 2,\\
o(b') = o((1,0)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{6}{\gcd(6,0)}\right) = \operatorname{lcm}(2,1) = 2
\end{aligned}$$

and $\langle a'\rangle \cap \langle b'\rangle = \langle (0,1)\rangle \cap \langle (1,0)\rangle = \{(0,0)\}$, we have $o(b') = o(b) < c = c'$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle\cap\langle b'\rangle|} = \frac{6(2)}{1} = 12$, which imply $r' = o(b' + \langle a'\rangle) = \frac{|H'|}{o(a')} = \frac{12}{6} = 2 = r$. By Corollary 3.3, $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1,2),(0,3))$ is isomorphic to $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((0,1),(1,0))$.

Example 3.10. Let $a = (0,1,2), a' = (1,0,1), b = (1,1,1), b' = (0,1,1)$ be in $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$. From Example 2.8, we have $r = o((1,1,1) + \langle (0,1,2)\rangle) = 2$ and $\lambda((0,1,2),(1,1,1)) = 4$, so

$$rb' = 2(0,1,1) = (0,0,2) = -4(1,0,1) = -\lambda(a,b)a'.$$

From

$$\begin{aligned}
c = o(a) = o((0,1,2)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,2)}\right) = \operatorname{lcm}(1,2,3) = 6,\\
c' = o(a') = o((1,0,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{2}{\gcd(2,0)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(2,1,3) = 6,\\
o(b) = o((1,1,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(2,2,3) = 6,\\
o(b') = o((0,1,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(1,2,3) = 6
\end{aligned}$$

and $\langle a'\rangle \cap \langle b'\rangle = \langle (1,0,1)\rangle \cap \langle (0,1,1)\rangle = \{(0,0,0),(0,0,1),(0,0,2)\}$, we have $o(b') = o(b) = c = c'$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle\cap\langle b'\rangle|} = \frac{6(6)}{3} = 12$, which imply $r' = o(b' + \langle a'\rangle) = \frac{|H'|}{o(a')} = \frac{12}{6} = 2 = r$. By Corollary 3.3, $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0,1,2),(1,1,1))$ is isomorphic to $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((1,0,1),(0,1,1))$.

Example 3.11. Let $a = (6,9), a' = (6,15), b = (12,18), b' = (12,6)$ be in $\mathbb{Z}_{36}\times\mathbb{Z}_{36}$. Then

$$\begin{aligned}
c = o(a) = o((6,9)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,6)}, \tfrac{36}{\gcd(36,9)}\right) = \operatorname{lcm}(6,4) = 12,\\
c' = o(a') = o((6,15)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,6)}, \tfrac{36}{\gcd(36,15)}\right) = \operatorname{lcm}(6,12) = 12,\\
o(b) = o((12,18)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,12)}, \tfrac{36}{\gcd(36,18)}\right) = \operatorname{lcm}(3,2) = 6,\\
o(b') = o((12,6)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,12)}, \tfrac{36}{\gcd(36,6)}\right) = \operatorname{lcm}(3,6) = 6,\\
\langle a\rangle \cap \langle b\rangle &= \langle (6,9)\rangle \cap \langle (12,18)\rangle = \{(0,0),(12,18),(24,0),(0,18),(12,0),(24,18)\},\\
\langle a'\rangle \cap \langle b'\rangle &= \langle (6,15)\rangle \cap \langle (12,6)\rangle = \{(0,0),(0,18)\}.
\end{aligned}$$

Since $|H| = \frac{o(a)\cdot o(b)}{|\langle a\rangle\cap\langle b\rangle|} = \frac{12(6)}{6} = 12$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle\cap\langle b'\rangle|} = \frac{12(6)}{2} = 36$, $r = o(b + \langle a\rangle) = \frac{|H|}{o(a)} = \frac{12}{12} = 1 \neq 3 = \frac{36}{12} = \frac{|H'|}{o(a')} = o(b' + \langle a'\rangle) = r'$. By Corollary 3.3, $C_{\mathbb{Z}_{36}\times\mathbb{Z}_{36}}((6,9),(12,18))$ is not isomorphic to $C_{\mathbb{Z}_{36}\times\mathbb{Z}_{36}}((6,15),(12,6))$.

Proposition 3.12. *Let $a, b \in G$, as a cyclic group of order n. Then*

(1) $o(a) = \frac{n}{\gcd(n,a)}$.

(2) $o(b + \langle a \rangle) = \frac{\gcd(n,a)}{\gcd(n,a,b)}$.

(3) $h = \frac{n \gcd(n,a,b)}{\gcd(n,a)\gcd(n,b)}$.

Proof. (1) comes from Corollary 3.8 (1). (2) is obtained from Corollary 3.8 (3) and remark after Theorem 1.4. (3) can be proved by the fact that $h = \frac{o(a)}{o(a+\langle b \rangle)}$. $\square$

Example 3.13. Let $a = 3, a' = 21, b = 5, b' = 55$ be in $\mathbb{Z}_{60}$. Then $o(b') = o(55) = \frac{60}{\gcd(60,55)} = 12 = \frac{60}{\gcd(60,5)} = o(5) = o(b) < c = o(3) = \frac{60}{\gcd(60,3)} = 20 = \frac{60}{\gcd(60,21)} = o(21) = c'$ and $r = o(5 + \langle 3 \rangle) = \frac{\gcd(60,3)}{\gcd(60,3,5)} = 3 = \frac{\gcd(60,21)}{\gcd(60,21,55)} = o(55 + \langle 21 \rangle) = r'$. From Lemma 2.2, since

$$15 = 3(5) = rb = \lambda(a,b)a = \lambda(a,b)3$$

for some $\lambda(a,b) \in \{0, 1, \ldots, 19 = c - 1\}$, $\lambda(a,b) = 5$. Thus,

$$rb' = 3(55) = 165 = 45 = 105 = 5(21) = \lambda(a,b)a'.$$

This shows that $C_{\mathbb{Z}_{60}}(3,5)$ is isomorphic to $C_{\mathbb{Z}_{60}}(21,55)$.

Example 3.14. Let $a = a' = 2, b = 9, b' = 15$ be in $\mathbb{Z}_{42}$. Then $o(b') = o(15) = \frac{42}{\gcd(42,15)} = 14 = \frac{42}{\gcd(42,9)} = o(9) = o(b) < c = c' = o(2) = \frac{42}{\gcd(42,2)} = 21$ and $r = o(9 + \langle 2 \rangle) = \frac{\gcd(42,2)}{\gcd(42,2,9)} = 2 = \frac{\gcd(42,2)}{\gcd(42,2,15)} = o(15 + \langle 2 \rangle) = r'$. From Lemma 2.2, since

$$18 = 2(9) = rb = \lambda(a,b)a = \lambda(a,b)2$$

for some $\lambda(a,b) \in \{0, 1, \ldots, 20 = c - 1\}$, $\lambda(a,b) = 9$. Thus,

$$rb' = 2(15) = 30 \neq \pm 18 = 9(2) = \pm\lambda(a,b)a'.$$

This shows that $C_{\mathbb{Z}_{42}}(2,15)$ is not isomorphic to $C_{\mathbb{Z}_{42}}(2,9)$.

References

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