Notes on Number Theory and Discrete Mathematics
Vol. 18, 2012, No. 3, 38–44

Generalized differential operators

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Abstract: This paper considers some properties of generalized differential operators by extending Chak and Schur derivatives as previously investigated by Leonard Carlitz. They are applied in the context of extended Laguerre polynomials.

Keywords: Rising and falling factorials, Binomial coefficients, Chak derivatives, Schur derivatives, Laguerre polynomials, *q*-series.

AMS Classification: 11B65, 11B39, 05A30.

1 Introduction

n numerous papers, Carlitz considered generalized versions of differential operators in the form of Chak derivatives defined by

$$D_z f(x) = \frac{f(zx) - f(x)}{(z-1)x} \tag{1.1}$$

Carlitz also studied properties associated with the Schur derivative

$$\Delta a_m = \frac{a_{m+1} - a_m}{p^{m+1}} \tag{1.2}$$

in which $\{a_m\}$ is a sequence and p is a prime number [1, 2].

In [10], we have also used these operators in association with Lah numbers arising from a search for some multisection formulas. The purpose of this paper is to investigate some properties of generalized differential operators, particularly in relation to Laguerre polynomials.

2 Rising and falling factorials

In a previous paper [11], falling and rising factorials were utilised in the following forms. The falling factorial, an *r*-permutation of *n* distinct objects, is given by

$$n^{\underline{r}} = P(n,r) = n(n-1)...(n-r+1) \tag{2.1}$$

and is such that

$$\nabla P(n,r) = P(n,r) - P(n-1,r) = rP(n-1,r-1), \tag{2.2}$$

Similarly, we showed for the rising factorial of *n*

$$n^{\bar{r}} = n(n+1)...(n+r-1) \tag{2.3}$$

that

$$\nabla n^{\bar{r}} = n^{\bar{r}} - (n-1)^{\bar{r}} = rn^{\overline{r-1}}. \tag{2.4}$$

This is a recurrence relation for $n^{\bar{r}}$, which is an r permutation of $n + r - 1$ objects, and which is related to the Stirling numbers [9]. Corresponding binomial coefficients were also considered, namely,

$$\binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r(r-1)...(r-r+1)} = \frac{n^{\underline{r}}}{r^{\underline{r}}} \tag{2.5}$$

in which $n^{\underline{r}}$ is the falling *r*-factorial of *n* and

$$C(n,r) = \frac{n^{\bar{r}}}{r^{\bar{r}}} \tag{2.6}$$

in which $n^{\bar{r}}$ is the rising *r*-factorial of *n*. Thus,

$$C(n,r) = \frac{n(n+1)...(n+r-1)}{r(r+1)...(r+r-1)}, \tag{2.7}$$

which is also suggested by the Gauss-Cayley form of the generalized binomial coefficient [8]. Here, it is proposed to illustrate and extend Carlitz' approach to cycles of binomial coefficients [4, 5].

3 Extended Laguerre polynomials

The search for a cycle of the type $\sum \left\langle {n \atop k} \right\rangle_a \binom{n}{k} x^k$, in which the rising binomial coefficient – type 1 is defined by [12]

$$\left\langle {n \atop k} \right\rangle_a = \frac{a^{\bar{n}}}{a^{\bar{k}} a^{\overline{n-k}}} \tag{3.1}$$

although it did finish up as

$$\sum \left\langle {n \atop k} \right\rangle_a \binom{n}{k} x^k = \sum \left\langle {n \atop k} \right\rangle_{a+1} \binom{n}{k} \frac{(-x)^k}{a!(a+1+n-k)^{\overline{k-a}}}$$

which is a form of the Laguerre polynomials for which Carlitz [6] has developed some generating functions. Carlitz also defined

$$L_n^{(a)}(x) = \sum_{k=0}^{n} (-1)^k \frac{(a+1)^{\overline{n}}}{(a+1)^{\overline{k}}\, k!(n-k)!} x^k \tag{3.2}$$

where

$$L_n(x) = n!\, L_n^{(0)}(x).$$

The ordinary Laguerre polynomials can be expressed as:

$$L_n(x) = n! \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^r}{r!}. \tag{3.3}$$

The equivalence of the two forms depends on the relation

$$n! = (n-k+a+1)^{\overline{k-a}}\,(a+1)^{\overline{n-k}}\, a!,$$

the proof of which follows. The right hand side of (2.3) equals

$$\begin{aligned} n(n-1)\ldots(n-k+a+1)(a+1)^{\overline{n-k}}\, a! \quad &= n(n-1)\ldots(n-k+a+1)(a+1)(a+2)\ldots(a+n-k)a! \\ &= n(n-1)\ldots(n-k+a+1)(n-k+a)\ldots(a+2)(a+1)a! \\ &= n!, \end{aligned}$$

as required. Thus,

$$\begin{aligned} = \sum_{k=0}^{n} (-1)^k \frac{(a+1)^{\overline{n}}}{(a+1)^{\overline{k}}\, k!(n-k)!} x^k \quad &= \sum_{k=0}^{n} \frac{(a+1)^{\overline{n}}}{(a+1)^{\overline{k}}\,(a+1)^{\overline{n-k}}} \frac{n!}{k(n-k)!} \frac{(-1)^k x^k}{a!(a+1+n-k)^{\overline{a}}} \\ &= \sum_{k=0}^{n} (-1)^k \frac{(a+1)^{\overline{n}}}{(a+1)^{\overline{k}}\, k!(n-k)!} x^k \\ &= L_n^{(a)}(x). \end{aligned}$$

The ordinary Laguerre polynomials are defined for n a positive integer and x a positive real number by the equation

$$\exp\left(\frac{-xt}{1-t}\right) = (1-t)\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!}.$$

This suggests a definition of Laguerre polynomials in terms of q-series. Accordingly, we define formally

$$\exp\left(\frac{-xt}{1-t}\right) = (1-t)\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{(z)_n}.$$

where the q-series

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n),$$

and we use z to emphasize the connection with the ordinary integers. Then

$$\begin{aligned} \exp(t) \quad &= \sum_{n=0}^{\infty} \frac{t^n}{(z)_n} \\ &= \prod_{n=0}^{\infty} (1-z^n t)^{-1} \end{aligned}$$

(see Carlitz [7]).

$$\begin{aligned}\exp\left(\frac{-xt}{1-t}\right) &= \sum_{n=0}^{\infty}(-xt)^n \frac{(1-t)^{-n}}{(z)_n} \\ &= \sum_{n=0}^{\infty}(-xt)^n \sum_{r=0}^{\infty}\binom{n+r-1}{r}\frac{t^r}{(z)_n} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n-1}{r}(-x)^{n-r}\frac{t^n}{(z)_{n-r}}\end{aligned}$$

Thus,

$$\begin{aligned}(1-t)\sum_{n=0}^{\infty}L_n(x)\frac{t^n}{(z)_n} &= \sum_{n=0}^{\infty}L_n(x)\frac{t^n}{(z)_n} - \sum_{n=0}^{\infty}L_n(x)\frac{t^n+1}{(z)_n} \\ &= \sum_{n=0}^{\infty}L_n(x)\frac{t^n}{(z)_n} - \sum_{n=0}^{\infty}L_{n-1}(x)\frac{t^n}{(z)_{n-1}}\end{aligned}$$

if we define $L_{-1}(x)=0.$ Now,

$$(z)_{n-1} = \frac{(z)_n}{1-z^n},$$

and so,

$$(1-t)\sum_{n=0}^{\infty}L_n(x)\frac{t^n}{(z)_n} = \sum_{n=0}^{\infty}\{L_n(x)-(1-z^n)L_{n-1}(x)\}\frac{t^n}{(z)_n}.$$

Equating coefficients of t^n we get

$$L_n(x)-(1-z^n)L_{n-1}(x) = \sum_{r=0}^{n}(-1)^{n-r}\binom{n-1}{r}(z_{n-r+1})_{r-1}x^{n-r}, \tag{3.4}$$

which is a recurrence relation for these extended Laguerre polynomials.

$$\begin{aligned}\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right) &= \sum_{n=0}^{\infty}(-xt)^n \frac{(1-t)^{-n-1}}{(z)_n} \\ &= \sum_{n=0}^{\infty}(-xt)^n \sum_{r=0}^{\infty}\binom{n+r}{r}\frac{t^r}{(z)_n} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(-x)^{n-r}\frac{(z)_n}{(z)_{n-r}}\frac{t^n}{(z)_n} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(-x)^{r}\frac{(z)_n}{(z)_r}\frac{t^n}{(z)_n}\end{aligned}$$

from the symmetry of the binomial coefficients. On equating coefficients of t^n we get

$$L_n(x) = \sum_{r=0}^{\infty}(-1)^r\binom{n}{r}(z_{r+1})_{n-r-1}x^r$$

as a definition of the extended Laguerre polynomials. The analogy is quite complete and can be seen more readily in the form

$$L_n(x) = (z)_n \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} \frac{x^r}{(z)_r}. \tag{3.5}$$

when compared to (3.3).

4 Generalized differential operators

The analogy between (3.3) and (3.5) and the differential equation for ordinary Laguerre polynomials

$$L_n(x) \ = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

suggests that we define

$$L_n(x) = e(x) D_{zx}^n \left(x^n \left(e(x)\right)^{-1}\right)$$

for the extended Laguerre polynomials and that we investigate the properties of the operator D_{zx}. It turns out that

$$D_{zx} x^n = (1 - z^n) x^{n-1.}$$

It is notationally convenient at this stage to use $\overline{q}_n$, the n-th reduced Fermatian of index q, instead of $(q)_n$, which is defined [1, 2] in terms of

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n < 0) \\ 1 & (n = 0), \\ 1 + q + q^2 + \ldots + q^{n-1} & (n > 0) \end{cases} \tag{4.1}$$

so that $\underline{1}_n = n,$ and $\underline{1}_n! = n!,$ where $\underline{q}_n! = \underline{q}_n \underline{q}_{n-1} \ldots \underline{q}_1.$

The relation between the Fermatian numbers and the q-series is actually quite close as expressed in some of the properties. For example, if we consider the equality of their appropriate binomial coefficients, then

$$\begin{aligned} \begin{bmatrix} n \\ r \end{bmatrix}_q &= \frac{\underline{q}_n!}{\underline{q}!\underline{q}_{n-r}!} \\ &= \frac{(1-q)^n (q)_n}{(1-q)^r (q)_r (1-q)^{n-r} (q)_{n-r}} \\ &= \frac{(q)_n}{(q)_r (q)_{n-r}}. \end{aligned}$$

We continue to use z instead of q to emphasize analogies with ordinary integers. Thus, we define

$$D_{xz} x^n = \underline{z}_n x^{n-1},$$

from which

$$D_{x1} x^n = n x^{n-1} = \frac{d}{dx} x^n.$$

Furthermore, we let

$$\begin{aligned} D_{zx}x^n &= (z^n)_1 x^{n-1} \\ &= (1-z^n)x^{n-1} \\ &= (1-z)D_{xz}x^n \end{aligned}$$

with $D_{zx}z = 0.$ Other properties follow.

If a is a constant, then

$$D_{xz}ax^n = aD_{xz}x^n,$$

and

$$D_{xz}f(y) = D_{yz}f(y)D_{xz}y,$$

which for $z = 1$ reduces to

$$D_x f(y) = D_y f(y) D_x y$$

as an expression of the ordinary chain rule for differentiation.

5 Concluding comments

Other properties can be readily developed, such as

$$D_{xz}y^n = \underline{z}_n y^{n-1}D_{xz}y,$$

and

$$D_{xz}(x^n + y^n) = D_{xz}x^n + D_{xz}y^n,$$

and for $u = u(x)$, $v = v(x)$, we have an analogue of Leibnitz' theorem for the n-th derivative of a product of two functions,

$$\begin{aligned} D^n uv &= \sum_{r=0}^{n} \binom{n}{r} D^r u D^{n-r} v, \\ D_{xz}^n uv &= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_z D_{xz}^r u D_{xz}^{n-r} v. \end{aligned}$$

The proof follows readily by induction on n.

References

[1] Carlitz, L. q-Bernoulli Numbers and Polynomials. *Duke Mathematical Journal*. Vol. 15, 1948, 987–1000.

[2] Carlitz, L. The Schur Derivative of a Polynomial. *Proceedings of the Glasgow Mathematical Association.* Vol. 1, 1953, 59–163.

[3] Carlitz, L. Expansions of q-Bernoulli Numbers. *Duke Mathematical Journal*. Vol. 25, 1958, 355–364.

[4] Carlitz, L. Sums of products of Multinomial Coefficients. *Elemente der Mathematik*. Vol. 18, 1963, 37–39.

[5] Carlitz, L. Multiple Sums and Generating Functions. *Collectanea Mathematica*. Vol. 17, 1965, 281–296.

[6] Carlitz, L. Some Generating Functions of Laguerre Polynomials. *Duke Mathematical Journal*. Vol. 35, 1968, 825–828.

[7] Carlitz, L. Generating Functions. *The Fibonacci Quarterly*. Vol. 7, 1969, 359–393.

[8] Carlitz, L., J. Riordan. Two Element Lattice Permutation Numbers and Their *q*-generalization. *Duke Mathematical Journal*. Vol. 31, 1964, 371–388.

[9] Riordan, J. A Note on a *q*-extension of Ballot Numbers. *Journal of Combinatorial Theory*. Vol. 4, 1968, 191–193.

[10] Shannon, A. G. Some Properties of Modified Lah Numbers. *Notes on Number Theory and Discrete Mathematics.* Vol. 7, 2001, No. 4, 125–131.

[11] Shannon, A. G. Some *q*-Binomial Coefficients Formed from Rising Factorials. *Notes on Number Theory and Discrete Mathematics.* Vol. 12, 2006, No. 1, 13–20.

[12] Shannon, A. G. Some Generalized Rising Binomial Coefficients. *Notes on Number Theory and Discrete Mathematics*. Vol. 13, 2007, No. 1, 25–30.