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Pellian sequences and squares

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Abstract: Elements of the Pell sequence satisfy a class of second order linear recurrence relations which interrelate a number of integer properties, such as elements of the rows of even and odd squares in the modular ring Z_4. Integer Structure Analysis of this yields multiple-square equations exemplified by primitive Pythagorean triples, the Hoppenot equation and the equation for a sphere centred at the origin. The structure breaks down for higher powered triples so that solutions are blocked. However, Euler's extension of Fermat's Last Theorem does not work as the structure does permit multiple power equations such as $a^5 + b^5 + c^5 + d^5 = e^5$.

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1 Introduction

It is well-known that any three consecutive odd integers, N, satisfy the 'Pellian-type' homogenous second order linear recurrence relation

$$N_{i+1} = 2N_i - N_{i-1}, \tag{1.1}$$

as do any three consecutive even integers.

We have previously shown that the elements of the rows of the tabulated representation Z_4 (Table 1) satisfy Pellian-type sequences [4] as do the geometric numbers [5]. Terr [8] has also considered Pellian-type sequences as part of "Pythagorean triple families".

The Pellian-type structure of rows of odd and even squares permits primitive Pythagorean triples to form, whereas higher powers lack such structure, so that triples of N^m, $m > 2$, cannot form. Obviously, Pellian-type sequences are useful structural features that can be used to analyse integer equations.

Here we consider the equation

$$a^2 + b^2 = r^2 - c^2 \tag{1.2}$$

which applies to a sphere of radius r whose centre is at the origin [1].

Row	$f(r)$	$4r_0$	$4r_1 + 1$	$4r_2 + 2$	$4r_3 + 3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1. Rows of modular ring Z_4

2 Equation of a sphere at the origin

In the primitive form of Equation (1.2), a, b, c and r cannot all have the same parity. Since the odd squares must belong to the class $\bar{1}_4$ and the even squares fall in $\bar{0}_4$, a and b must have opposite parity, with r odd and c even. The elements of the rows of odd squares $\{R_1\}$ and even squares $\{R_0\}$ satisfy an inhomogeneous form of (1.1), namely

$$N_{i+1} = 2N_i - N_{i-1} + 2. \tag{2.1}$$

Thus, in Equation (1.2)

$$a^2 = 4R_1 + 1,\ b^2 = 4R_0,\ r^2 = 4R_1^{/} + 1,\ c^2 = 4R_0^{/},$$

so that

$$R_1 + R_0 = R_1^{/} - R_0^{/}. \tag{2.2}$$

If we take the rows as the $(i+1)^{\text{th}}$, then Equation (2.2) has the form

$$2\left((R_1)_i + (R_0)_k + 2\right) - \left((R_1)_{i-1} + (R_0)_{k-1}\right) = 2\left((R_1^{/})_j - (R_0^{/})_l\right) - \left((R_1^{/})_{j-1} - (R_0^{/})_{l-1}\right) \tag{2.3}$$

which shows the compatibility of the left and right hand sides of Equation (1.2). The values of a, b, r and c will be restricted so that the right end digits (REDs) are compatible. This yields seven possible combinations (Table 2).

If we take the second example for Set 4 in Table 2, then we get the following values:

$$(R_1)_i = 240,\quad (R_0)_k = 196,\quad (R_1)_{i-1} = 210,\quad (R_0)_{i-1} = 169,$$
$$(R_1^{/})_j = 1640,\quad (R_0^{/})_l = 1156,\quad (R_1^{/})_{j-1} = 1560,\quad (R_0^{/})_{l-1} = 1089,$$

and so the two sides of Equation (2.3) equal 497.

No.	$(a^2)^*$	$(b^2)^*$	$(r^2)^*$	$(c^2)^*$	Examples for			
					a	*b*	*r*	*c*
1	1	0	1	0	1	50	51	10
2	1	0	5	4	11	10	15	2
3	5	0	1	6	5	20	21	4
4	9	0	9	0	23	10	27	10
					33	30	83	70
5	9	0	5	6	33	10	35	6
6	1	6	1	4	21	6	31	22
7	9	4	9	6	3	2	13	4

Table 2. Possible combinations for Equation (1.2)

3 Multiple-squares equations

Suppose that we want to check the equation

$$a^2 + b^2 + c^2 + d^2 = e^2. \tag{3.1}$$

Parity considerations show that three of the squares on the left hand side will be even and fourth will be odd, so that e^2 is odd. The row equation analogous to (2.3) is then

$$\begin{aligned} &2\left((R_0)_i + (R_0')_k + (R_0'')_l + (R_1)_m + 3\right) - \left((R_0)_{i-1} + (R_0')_{k-1} + (R_0'')_{l-1} + (R_1)_{m-1}\right) + 2 \\ &= 2(R_1')_n - (R_1')_{n-1} + 2. \end{aligned} \tag{3.2}$$

The left hand side of (3.2) is compatible with the right hand side so solutions should occur. For example, the solution set

$$\{a,b,c,d,e\} = \{6,16,10,7,21\}$$

yields the rows

$$R_i: 4,49,16,6,90;\ \ R_{i-1}: 1,36,9,2,72,$$

so that the left and right hand sides of Equation (3.2) each equal 110.

The main compatibility factor is obviously the fact that the rows of both odd and even squares are elements of the same Pellian sequence (Equation (2.1)). Higher powers do not have this compatibility so that triples cannot form with these powers [4]. Sums of squares may be reduced to a square if they meet the pPt criteria [2, 6]. Thus, Equation (3.1) may be reduced to a pPt.

The Hoppenot equation [6] is another example of multiple-square relationships. These sums are a function of the triangular and pyramidal numbers, which, in turn, are elements of Pellian sequences [5].

4 Final comments

In Z_4, all even powers $\in \{\bar{0}_4, \bar{1}_4\}$ and odd powers $\in \{\bar{0}_4, \bar{1}_4, \bar{3}_4\}$. Thus, for a system such as

$$a^5 + b^5 + c^5 + d^5 = e^5, \tag{4.1}$$

the left hand side may have even numbers of T_4 or $\bar{3}_4$ (that is, 2, 4, etc.) or a mixed $\{\bar{1}_4, \bar{3}_4\}$ structure [3]. This yields an increase in degrees of freedom so that solutions may occur, such as

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5. \tag{4.2}$$

For the triple, only one odd class occurs on the left hand side so that compensation for the row structure cannot occur. On the other hand, the limited distribution of even powers in only two classes restricts solutions for

$$a^4 + b^4 + e^4 = d^4, \tag{4.3}$$

so that solutions are rare and the components large; for instance [3]:

$$95800^4 + 217519^4 + 414560^4 = 422481^4. \qquad (4.3)$$

An analysis of Euler's extension of Fermat's Last Theorem has been published previously [3]. In general, integer structure does not permit power triples. However, for squares the rows of even and odd squares follow the same Pellian structure and this allows triples to form. Further research relevant to this would be to extend Melham's work with Fibonacci and Lucas numbers [7] to numbers which satisfy Pellian-type recurrence relations.

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