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The common minimal equitable dominating signed graphs

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Abstract: In this paper, we define the common minimal equitable dominating signed graph of a given signed graph and offer a structural characterization of common minimal equitable dominating signed graphs. In the sequel, we also obtained switching equivalence characterization: $\overline{\Sigma} \sim CMED(\Sigma)$, where $\overline{\Sigma}$ are $CMED(\Sigma)$ are complementary signed graph and common minimal equitable dominating signed graph of Σ respectively.
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1 Introduction

For standard terminology and notation in graph theory we refer Harary [7] and Zaslavsky [30] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Signed graphs, in which the edges of a graph are labelled positive or negative, have developed many applications and a flourishing literature (see [30]) since their first introduction by Harary in 1953 [8]. Their natural extension to multisigned graphs, in which each edge gets an n-tuple of signs—that is, the sign group is replaced by a direct product of sign groups—has received slight attention, but the further extension to gain graphs (also known as voltage graphs), which have edge labels from an arbitrary group such that reversing the edge orientation inverts the label, have been well studied [30]. Note that in a multisigned group every element is its own inverse, so the question of edge reversal does not arise with multisigned graphs.

A *signed graph* $\Sigma = (\Gamma, \sigma)$ is a graph $\Gamma = (V, E)$ together with a function $\sigma : E \to \{+, -\}$, which associates each edge with the sign + or -. In such a signed graph, a subset A of $E(\Gamma)$ is said to be *positive* if it contains an even number of negative edges, otherwise is said to be *negative*. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced* [8] if in every cycle the product of the edge signs is positive. Σ is antibalanced [9] if in every even (odd) cycle the product of the edge signs is positive (resp., negative); equivalently, the negated signed graph $-\Sigma = (\Gamma, -\sigma)$ is balanced. A *marking* of Σ is a function $\mu : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking μ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\mu(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking μ of Σ is called *canonical marking* of Σ. In a signed graph $\Sigma = (\Gamma, \sigma)$, for any $A \subseteq E(\Gamma)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Proposition 1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [8]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$. (Sampathkumar [14]).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Proposition 1. For more new notions on signed graphs refer the papers ([11], [12], [15], [16], [18]–[26]).

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [30].

If $\mu : V(\Gamma) \to \{+, -\}$ is *switching function*, then *switching* of the signed graph $\Sigma = (\Gamma, \sigma)$ by μ means changing σ to σ^μ defined by:

$$\sigma^\mu = \mu(u)\sigma(uv)\mu(v).$$

The signed graph obtained in this way is denoted by Σ^μ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *isomorphic*, written as $\Sigma_1 \cong \Sigma_2$ if there exists a graph isomorphism $f : \Gamma_1 \to \Gamma_2$ (that is a bijection $f : V(\Gamma_1) \to V(\Gamma_2)$ such that if uv is an edge in Γ_1 then $f(u)f(v)$ is an edge in

Γ_2) such that for any edge $e \in E(\Gamma_1)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $\Sigma_1 = (\Gamma_1, \sigma_1)$ *switches* to a signed graph $\Sigma_2 = (\Gamma_2, \sigma_2)$ (or that Σ_1 and Σ_2 are *switching equivalent*) written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $\Sigma_1^\mu \cong \Sigma_2$. Note that $\Sigma_1 \sim \Sigma_2$ implies that $\Gamma_1 \cong \Gamma_2$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [27]) or *cycle isomorphic* (see [29]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (See [29]):

Proposition 2. **(T. Zaslavsky** [29]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [17], the authors introduced the switching and cycle isomorphism for signed digraphs.

2 Common minimal equitable dominating signed graphs

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [4] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [13] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [2] in 1958. Berge wrote a book on graph theory, in which he introduced the “coefficient of external stability”, which is now known as the domination number of a graph. Oystein Ore [10] introduced the terms “dominating set” and “domination number” in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [28]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [3] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

A subset D of $V(\Gamma)$ is called an *equitable dominating set* of a graph Γ, if for every $v \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(\Gamma)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of Γ. An equitable dominating set D is *minimal*, if for any vertex $u \in D$, $D - \{u\}$ is not a equitable dominating set of Γ. This concept was introduced by Deepak et al. [5].

In [5], the authors introduced a new class of intersection graphs in the field of domination theory. The common minimal equitable dominating graph is denoted by $CMED(\Gamma)$ is the graph which has the same vertex set as Γ with two vertices are adjacent if and only if there exist minimal equitable dominating in Γ containing them.

Motivated by the existing definition of complement of a signed graph, we extend the notion of common minimal equitable dominating graphs to signed graphs as follows: The *common minimal equitable dominating signed graph* $CMED(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $CMED(\Gamma)$ and sign of any edge uv in $CMED(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called common minimal equitable dominating signed graph, if $\Sigma \cong CMED(\Sigma')$ for some signed graph Σ'. In the following section, we shall present a characterization of common minimal equitable dominating signed graphs. The purpose of this paper is to initiate a study of this notion.

We now give a straightforward, yet interesting, property of common minimal equitable dominating signed graphs.

Proposition 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its common minimal equitable dominating signed graph $CMED(\Sigma)$ is balanced.*

Proof. Since sign of any edge uv in $CMED(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ, by Proposition 1, $CMED(\Sigma)$ is balanced. □

For any positive integer k, the k^{th} iterated common minimal equitable dominating signed graph $CMED(\Sigma)$ of Σ is defined as follows:

$$CMED^0(\Sigma) = \Sigma, CMED^k(\Sigma) = CMED(CMED^{k-1}(\Sigma))$$

Corollary 4. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and any positive integer k, $CMED^k(\Sigma)$ is balanced.*

In [5], the authors characterized graphs for which $CMED(\Gamma) \cong \overline{\Gamma}$.

Proposition 5. **(G. Deepak et al. [5])** *For any graph $\Gamma = (V, E)$, $CMED(\Gamma) \cong \overline{\Gamma}$ if and only if every minimal equitable dominating set of Γ is independent.*

We now characterize signed graphs whose common minimal equitable dominating signed graphs and complementary signed graphs are switching equivalent.

Proposition 6. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\Sigma} \sim CMED(\Sigma)$ if and only if every minimal equitable dominating set of Γ is independent.*

Proof. Suppose $\overline{\Sigma} \sim CMED(\Sigma)$. This implies, $\overline{\Gamma} \cong CMED(\Gamma)$ and hence by Proposition 5, every minimal equitable dominating set of Γ is independent.

Conversely, suppose that every minimal equitable dominating set of Γ is independent. Then $\overline{\Gamma} \cong CMED(\Gamma)$ by Proposition 5. Now, if Σ is a signed graph with every minimal equitable dominating set of underlying graph Γ is independent, by the definition of complementary signed graph and Proposition 3, $\overline{\Sigma}$ and $CMED(\Sigma)$ are balanced and hence, the result follows from Proposition 2. □

Proposition 7. *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their common minimal equitable dominating signed graphs are switching equivalent.*

Proof. Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Proposition 3, $CMED(\Sigma_1)$ and $CMED(\Sigma_2)$ are balanced and hence, the result follows from Proposition 2. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [9] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 6 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 8. *For any signed graph* $\Sigma = (\Gamma, \sigma)$*,* $\overline{\eta(\Sigma)} \sim CMED(\Sigma)$ *(or* $\overline{\Sigma} \sim CMED(\eta(\Sigma))$*) if, and only if, every minimal equitable dominating set of* Γ *is independent.*

Corollary 9. *For any signed graph* $\Sigma = (\Gamma, \sigma)$*,* $\overline{\eta(\Sigma)} \sim CMED(\eta(\Sigma))$ *if, and only if, every minimal equitable dominating set of* Γ *is independent.*

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $CMED(\Sigma)$ is balanced (Proposition 3). We now examine, the conditions under which negation of $CMED(\Sigma)$ is balanced.

Proposition 10. *Let* $\Sigma = (\Gamma, \sigma)$ *be a signed graph. If* $CMED(\Gamma)$ *is bipartite then* $\eta(CMED(\Sigma))$ *is balanced.*

Proof. Since, by Proposition 3, $CMED(\Sigma)$ is balanced, each cycle C in $CMED(\Sigma)$ contains even number of negative edges. Also, since $CMED(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $CMED(\Sigma)$ is also even. Hence $\eta(CMED(\Sigma))$ is balanced. □

3 Characterization of common minimal equitable dominating signed graphs

The following result characterizes signed graphs which are common minimal equitable dominating signed graphs.

Proposition 11. *A signed graph* $\Sigma = (\Gamma, \sigma)$ *is a common minimal equitable dominating signed graph if and only if* Σ *is balanced signed graph and its underlying graph* Γ *is a* $CMED(\Gamma)$*.*

Proof. Suppose that Σ is balanced and its underlying graph Γ is a common minimal equitable dominating graph. Then there exists a graph Γ' such that $CMED(\Gamma') \cong \Gamma$. Since Σ is balanced, by Proposition 2, there exists a marking μ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $CMED(\Sigma') \cong \Sigma$. Hence Σ is a common minimal equitable dominating signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a common minimal CN-dominating signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $CMED(\Sigma') \cong \Sigma$. Hence by Proposition 3, Σ is balanced. □

Problem 12. *Characterize signed graphs for which* $\overline{\Sigma} \cong CMED(\Sigma)$.

References

[1] Abelson, R. P., M. J. Rosenberg, Symbolic psychologic: A model of attitudinal cognition, *Behav. Sci.*, Vol. 3, 1958, 1–13.

[2] Berge, C., *Theory of Graphs and its Applications*, Methuen, London, 1962.

[3] Cockayne, E. J., S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks*, Vol. 7, 1977, 247–261.

[4] De Jaenisch, C. F., *Applications de l'Analyse mathematique an Jen des Echecs*, 1862.

[5] Deepak, G., N. D. Soner, A. Alwardi, Vertex minimal and common minimal equitable dominating graphs, *Int. J. Contemp. Math. Sciences*, Vol. 7, 2012, No. 10, 499–505.

[6] Easley, D., J. Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*, Cambridge University Press, 2010.

[7] Harary, F., *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[8] Harary, F., On the notion of balance of a signed graph, *Michigan Math. J.*, Vol. 2, 1953, 143–146.

[9] Harary, F., Structural duality, *Behav. Sci.*, Vol. 2, 1957, No. 4, 255–265.

[10] Ore, O., Theory of Graphs. *Amer. Math. Soc. Colloq. Publ.*, Vol. 38, 1962.

[11] Rangarajan, R., P. Siva Kota Reddy, The edge C_4 signed graph of a signed graph, *Southeast Asian Bulletin of Mathematics*, Vol. 34, 2010, No. 6, 1077–1082.

[12] Rangarajan, R., M. S. Subramanya, P. Siva Kota Reddy, Neighborhood signed graphs, *Southeast Asian Bulletin of Mathematics*, Vol. 36, 2012, No. 3, 389–397.

[13] Rouse Ball, W. W., *Mathematical Recreation and Problems of Past and Present Times*, 1892.

[14] Sampathkumar, E., Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, Vol. 7, 1984, No. 3, 91–93.

[15] Sampathkumar, E., P. Siva Kota Reddy, M. S. Subramanya, Directionally n-signed graphs, *Ramanujan Math. Soc., Lecture Notes Series (Proc. Int. Conf. ICDM 2008)*, Vol. 13, 2010, 155–162.

[16] Sampathkumar, E., P. Siva Kota Reddy, M. S. Subramanya, Directionally n-signed graphs-II, *International J. Math. Combin.*, Vol. 4, 2009, 89–98.

[17] Sampathkumar, E., M. S. Subramanya, P. Siva Kota Reddy, Characterization of line sidigraphs, *Southeast Asian Bulletin of Mathematics*, Vol. 35, 2011, No. 2, 297–304.

[18] Siva Kota Reddy, P., M. S. Subramanya, Note on path signed graphs, *Notes on Number Theory and Discrete Mathematics*, Vol. 15, 2009, No. 4, 1–6.

[19] Siva Kota Reddy, P., S. Vijay, V. Lokesha, n^{th} Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, Vol. 12, 2009, No. 3, 307–313.

[20] Siva Kota Reddy, P., t-Path Sigraphs, *Tamsui Oxford J. of Math. Sciences*, Vol. 26, 2010, No. 4, 433–441.

[21] Siva Kota Reddy, P., E. Sampathkumar, M. S. Subramanya, Common-edge signed graph of a signed graph, *J. Indones. Math. Soc.*, Vol. 16, 2010, No. 2, 105–112.

[22] Siva Kota Reddy, P., B. Prashanth, T. R. Vasanth Kumar, Antipodal signed directed Graphs, *Advn. Stud. Contemp. Math.*, Vol. 21, 2011, No. 4, 355–360.

[23] Siva Kota Reddy, P., B. Prashanth, The Common Minimal Dominating Signed Graph, *Trans. Comb.*, Vol. 1, 2012, No. 3, 39–46.

[24] Siva Kota Reddy, P., B. Prashanth, $\mathcal{S}$-Antipodal signed graphs, *Tamsui Oxford J. of Inf. Math. Sciences*, Vol. 28, 2012, No. 2, 165–174.

[25] Siva Kota Reddy, P., S. Vijay, The super line signed graph $\mathcal{L}_r(S)$ of a signed Graph, *Southeast Asian Bulletin of Mathematics*, Vol. 36, 2012, No. 6, 875–882.

[26] Siva Kota Reddy, P.,d B. Prashanth, Note on Minimal Dominating Signed Graphs, *Bull. of Pure & Appl. Math.*, Vol. 7, 2013, No. 1, to appear.

[27] Soz$\acute{a}$nsky, T., Enueration of weak isomorphism classes of signed graphs, *J. Graph Theory*, Vol. 4, 1980, No. 2, 127–144.

[28] Yaglom, A. M., I. M. Yaglom. Challenging mathematical problems with elementary solutions. Volume 1: *Combinatorial Analysis and Probability Theory*, 1964.

[29] Zaslavsky, T., Signed graphs, *Discrete Appl. Math.*, Vol. 4, 1982, No. 1, 47–74.

[30] Zaslavsky, T., A mathematical bibliography of signed and gain graphs and allied areas, VIII Edition, *Electron. J. Combin.*, Dynamic Surveys in Combinatorics (1998), No. DS8. Eighth ed. (2012).