Solving algebraic equations with Integer Structure Analysis

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Abstract: A new alternative method for solving algebraic equations is expounded. Integer Structure Analysis is used with an emphasis on parity, right-end-digits of the components and the modular ring $\mathbb{Z}_5$.

Keywords: Modular rings, Quadratic equations, Cubic equations, Simultaneous equations, Complex roots, Integer Structure Analysis.

AMS Classification: 11A07.

1 Introduction

Many methods exist for finding real and complex solutions of algebraic equations such as quadratic, cubic and simultaneous equations with two and three variables [1, 3]. Integer Structure Analysis (ISA) can be used with these traditional methods to illuminate some of the related underlying number theoretic foundations [6]. In this paper, we provide examples of various types of equations using the modular ring $\mathbb{Z}_5$ and right-end-digits (REDs) (Table 1). In this ring each class has a characteristic RED structure that simplifies analysis.

<table>
<thead>
<tr>
<th>Row</th>
<th>$f(r)$</th>
<th>$5r_0$</th>
<th>$5r_1 + 1$</th>
<th>$5r_2 + 2$</th>
<th>$5r_3 + 3$</th>
<th>$5r_4 + 4$</th>
</tr>
</thead>
<tbody>
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<td>$\overline{3}_5$</td>
<td>$\overline{4}_5$</td>
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</tr>
<tr>
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<td>0</td>
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<td>2</td>
<td>3</td>
<td>4</td>
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</table>

(table continues)
Table 1. Rows of $Z_5$

<table>
<thead>
<tr>
<th>Row</th>
<th>$f(r)$</th>
<th>$5r_0$</th>
<th>$5r_1 + 1$</th>
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<th>$5r_3 + 3$</th>
<th>$5r_4 + 4$</th>
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</tr>
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<td>51</td>
<td>52</td>
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</tr>
</tbody>
</table>

2 Quadratic equations

In the following sections we solve some simple algebraic equations with an ISA approach to show how it sheds some light on the underlying number theoretic structure.

1. \[5x^2 - 17x + 14 = 0.\] (2.1)

If $x$ is odd then $x^* = 7$, and if $x$ is even, then $x^* = 2$, so

\[x = 5r_2 + 2.\] (2.2)

Substituting (2.2) into (2.1) yields

\[25r_2^2 - 3r_2 = 0.\] (2.3)

Thus $r_2 = 0$ or $-3/25$, and so $x = 2$ or $7/5$.

2. \[f(x) \equiv x^2 - 11x + 30 = 0.\] (2.4)

$x$ can be odd or even.

<table>
<thead>
<tr>
<th>$x^*$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>$x^*$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x^2)^*$</td>
<td>1</td>
<td>9</td>
<td>5</td>
<td>9</td>
<td>1</td>
<td>$(x^2)^*$</td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$(-11x)^*$</td>
<td>-1</td>
<td>-3</td>
<td>-5</td>
<td>-7</td>
<td>-9</td>
<td>$(-11x)^*$</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
<td>-6</td>
<td>-8</td>
</tr>
<tr>
<td>$(30)^*$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(30)^*$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(f(x))^*$</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>$(f(x))^*$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 2: Solution structure for Equation (2.4)

Thus $x^* = 1$ or $5$ and $x^* = 0$ or $6$, but $x \neq 1$ or $0$, and so $x = 5$ or $6$. Alternatively, use $Z_5$, take $x = 5r + a$ and substitute into (2.4). If $a = 6$, the constant $30$ is eliminated and we obtain

\[25r^2 + 5r = 0.\] (2.5)

Thus, $r = 0$ or $-1/5$ which gives $x = 6$ or $5$. 

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3 Cubic equations

1. \( x^3 + 4x - 5 = 0. \) \hspace{1cm} (3.1)

\( x \) must be odd.

<table>
<thead>
<tr>
<th>( x^* )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (x^3)^* )</td>
<td>1</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>( (4x)^* )</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>( (-5)^* )</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>( (f(x))^* )</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Structure for Equation (3.1)

Thus \( x^* = 1, 5 \) or \( 9 \), but 5 and 9 are too large, so \( x^* = 1 \), and

\( x = 5r_1 + 1. \) \hspace{1cm} (3.2)

Substituting into (3.1) yields \( r_1 = 0 \) or

\[ 25r_1^2 + 15r_1 + 7 = 0. \] \hspace{1cm} (3.3)

Then using \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), we get

\[ \eta = \frac{-3 \pm \sqrt{19i}}{10} \] \hspace{1cm} (3.4)

so that

\[ x = 1, \frac{-1 \pm \sqrt{19i}}{2} \]

2. \( x^3 + 5x^2 + 3x - 9 = 0. \) \hspace{1cm} (3.5)

<table>
<thead>
<tr>
<th>( x^* )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (5x^2)^* )</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>( (3x)^* )</td>
<td>±3</td>
<td>±9</td>
<td>±5</td>
<td>±1</td>
<td>±7</td>
</tr>
<tr>
<td>( (x^3)^* )</td>
<td>±1</td>
<td>±7</td>
<td>±5</td>
<td>±3</td>
<td>±9</td>
</tr>
<tr>
<td>( (-q)^* )</td>
<td>-9</td>
<td>-9</td>
<td>-9</td>
<td>-9</td>
<td>-9</td>
</tr>
<tr>
<td>( (f(x))^* )</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4: Structure for Equation (3.5)

From Table 4, \( x^* = 1 \) or \( 7 \), but 7 is too large so

\( x = 5r_1 + 1. \) \hspace{1cm} (3.6)

Substituting into (3.5) gives \( r_1 = 0 \) and

\[ 25r_1^2 + 40r_1 + 16 = 0. \] \hspace{1cm} (3.7)

Thus \( r_1 = 0 \) or \( -4/5 \), and so \( x = 1 \) or \( -3 \). When \( x \) is negative, \( x^* = -3 \), and so \( x = 1, -3, -3 \). For other non-conventional number theoretic approaches to cubic equations see [2, 4, 5, 6].
4 Simultaneous equations

4.1 Two variables

1. \[3x + 7y = 27\]  
   \[5x + 2y = 16\]  
   \(x\) must be even from (4.2), so from (4.1) \(y\) must be odd. Since \((5x)^* = 0\) and \(y\) is odd, only \(y^* = 3\) or \((2y)^* = 6\). Thus \(y = 3, 13, 23, ...\) but only \(y = 3\) fits, so that the solution is \(x = 2, y = 3\).

2. \[x^2 + 4y^2 + 80 = 15x + 30y\]  
   \(xy = 6\)  
   From (4.4) \(x = 6/y\) which, when substituted into (4.3), gives \[4y^4 - 30y^3 + 80y^2 - 90y + 36 = 0\]  
The RED right hand side is zero, so \((4y^4 + 36)^* = 0\). Hence, \(y^* = 1, 2, 3\) which fits (4.5) and so yields \(x = 6, 3, 2\). But Equation (4.3) should have 4 roots, so we substitute (4.4) into (4.3) to get \[x^4 - 15x^3 + 144 = 0\]  
Thus, \((x^4 - 15x^3 + 144)^* = 0\). (4.7)

Apart from \(x = 6, 3, 2\), \(x^* = 4\) yields \((6 - 0 + 4)^* = 0\), so that \(x = 4\) and \(y = 3/2\). Hence, \(\{(x, y)\} = \{(6,1),(3,2),(2,3),(4,3/2)\}\).

3. \[3x^2 - 5y^2 = 28\]  
   \[3xy - 4y^2 = 8\]  
   From (4.8) we see that \(x\) and \(y\) must have the same parity, and then from (4.9) that they must both be even. Thus \((x^2)^* = 6\) only, but \((y^2)^* = 0, 4\) or 6. Hence \(x = \pm 4\) or \(\pm 6\), but \(y = 0, \pm 2, \pm 8, \pm 4\) or \(\pm 6\). Since \(y < x, y^* \neq 6\) or 8 and \(y \neq 0\). Therefore,

   \[x = \pm 4, \pm 6\]  and \(y = \pm 2, \pm 4\).

4.2 Three variables

1. \[x + 2y + 2z = 11\]  
   \[2x + y + z = 7\]  
   \[3x + 4y + z = 14\]  
   From (4.10) \(x\) is odd; from (4.12) \(z\) is odd, and so from (4.11) \(y\) is even. If we then subtract (4.11) from (4.12) we get \[x + 3y = 7\].  
The \(x^*, y^*\) values which satisfy (4.13) are shown in Table 5.
Thus,

\[ x = 1, y = 2, z = 3. \]

Note that if \( x = 5r + a \) and \( y = 5r + b \), then \( a = 1 \) and \( b = 2 \), so that the solution classes are \( x \in \mathbb{T}_5, y \in \mathbb{Z}_5, z \in \mathbb{Z}_5 \).

2. \[ \begin{align*} x + 4y + 3z &= 17 \quad (4.14) \\ 3x + 3y + z &= 16 \quad (4.15) \\ 2x + 2y + z &= 11 \quad (4.16) \end{align*} \]

From (4.16) \( z \) is odd, so from (4.14) \( x \) is even, and from (4.15) \( y \) is odd. If we then subtract (4.16) from twice (4.14) we get

\[ 6y + 5z = 23 \quad (4.17) \]

Thus \( y^* = 3 \) and \( y = 3 \) fits so solution set is

\[ x = 2, y = 3, z = 1. \]

3. \[ \begin{align*} 2x + 3y + 4z &= 20 \quad (4.18) \\ 3x + 4y + 5z &= 26 \quad (4.19) \\ 3x + 5y + 6z &= 31 \quad (4.20) \end{align*} \]

We subtract (4.19) from (4.20) to get

\[ y + z = 5 \quad (4.21) \]

From (4.18), \( y \) is even, so from (4.21) \( z \) is odd and from (4.20) \( x \) is odd. Table 6 shows the values for \( y^*, z^* \) which are compatible with (4.21).

<table>
<thead>
<tr>
<th>( y^* )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^* )</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>9</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 6: \( y^*, z^* \) values for (4.21)

Hence, \( y \neq 6, 8 \) (too large) and \( y \neq 0, 4 \) from Equations (4.18) and (4.20). Thus the solution set is

\[ x = 1, y = 2, z = 3. \]
5 Complex roots

1. \[ x^4 + x^3 - 6x^2 - 15x - 9 = 0. \] (5.1)

Let \( x = 5r + a \). Then, if the constant in (5.1) is to be cancelled when this value of \( x \) is substituted, \( a = 3 \), which yields \( r = 0 \) and \( x = 3 \). Equation (5.1) now reduces to

\[ x^3 + 4x^2 + 6x + 3 = 0. \] (5.2)

Again let \( x = 5r + a \) and substitute into (5.2). Elimination of the constant occurs when \( a = -1 \), and since \( r = 0 \) is a root then \( x = -1 \). Equation (5.2) then reduces to

\[ x^2 + 3x + 3 = 0. \] (5.3)

We then use \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) again to get the complex roots, with the complete solution set

\[ x = -1, 3, \frac{-3 \pm \sqrt{3}i}{2} \]

Note that \( 5r - 1 = (5(s - 1) + 4) \in \mathbb{Z}_s \).

2. \[ x^4 - 3x^3 + 12x - 16 = 0. \] (5.4)

Let \( x = 5r + a \). Elimination of the constant yields \( a = \pm 2 \) which, when substituted into (5.4) both reduce the latter to

\[ x^3 - x^2 - 2x + 8 = 0. \] (5.5)

and \( r = 0 \) is one solution so that \( x = \pm 2 \).

If \( x = 5r - 2 \) (to eliminate 8), then (5.5) becomes

\[ 25r^2 - 35r + 14 = 0. \] (5.6)

so that

\[ r = \frac{7 \pm \sqrt{7}i}{10}, \] (5.7)

or \( x = 5r - 2 = \frac{3 \pm \sqrt{7}i}{2} \). The solution set is then \( \{\pm 2, \frac{3 \pm \sqrt{7}i}{2} \} \).

3. \[ x^3 + 1 = 0. \] (5.8)

With \( x = 5r - 1 \) the constant is eliminated. Substitution into (5.8) yields \( r = 0 \) and

\[ 25r^2 - 15r + 3 = 0. \] (5.9)

or

\[ r = \frac{3 \pm \sqrt{3}i}{10}, \] (5.10)

so that the solution set is \( \{-1, (3 \pm \sqrt{3}i)/2 \} \). (Note that \( 5r - 2 = (5(s - 1) + 3) \in \mathbb{Z}_s \) in row \( (s - 1) \).)
6 Final comments

Students should find the approach outlined here in the examples an interesting alternative to what can sometimes degenerate into a ‘symbol-shoving’ exercise. There is also potential for project work which extends this type of analysis to other types of equations and using other modular rings [6].

References