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Generalized Hurwitz series

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Abstract: Properties of generalized Hurwitz series are developed here in the framework of Fermatian numbers. These properties include derivatives in the Fontené–Jackson calculus which results in another solution of Ward's Staudt–Clausen problem

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1 Introduction

The purpose of this paper is to consider some properties of series defined formally as

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{\underline{z}_n!} \tag{1.1}$$

in which the a_n are arbitrary integers and the $\underline{z}_n$ are Fermatian numbers [30]. We shall call such series a generalized Hurwitz series (GH–series). When $z = 1$, we get the ordinary Hurwitz series [7]. This will involve references to Ward's generalized Staudt–Clausen problem [38] and related work by Carlitz [5, 16].

If we consider another GH–series

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{\underline{z}_n!} \tag{1.2}$$

then the product of (1.1) and (1.2) is another GH-series [26]:

$$\sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} a_r b_{n-r} \right) \frac{t^n}{\underline{z}_n!} \tag{1.3}$$

in which

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{\underline{z}_n!}{\underline{z}_r! \underline{z}_{n-r}!}$$

as defined in (2.10 to (2.4) below.

2 Fermatian numbers

Fermatian numbers may be defined in terms of real numbers z such that

$$\underline{z}_n = \begin{cases} -z^n \underline{z}_n & (n < 0) \\ 1 + z + z^2 + \ldots + z^{n-1} & (n > 0) \\ 1 & (n = 0) \end{cases} \tag{2.1}$$

so that

$$\underline{1}_n = n \tag{2.2}$$

and

$$\underline{1}_n! = n!, \tag{2.3}$$

where

$$\underline{z}_n! = \underline{z}_n \underline{z}_{n-1} \ldots \underline{z}_1 \tag{2.4}$$

Some properties of these numbers may be found in [35].

Carlitz and Moser [21] examined some of the Fermatian properties by giving all the possible factorizations of $\underline{x}_n$ into its product of C-polynomials over the field of rational numbers, where the C–polynomial of A is defined by

$$A(x) = x^{a_1} + x^{a_2} + \ldots + x^{a_k}$$

for

$$A = \{a_1, a_2, \ldots, a_k\}$$

an ordered set of non-negative integers. A particularly interesting result of Carlitz and Moser is that if $f(n)$ denotes the number of factorizations

$$\underline{x}_n = A(x)B(x),$$

where $A(x)$, $B(x)$ are C–polynomials, then

$$2f(n) = \sum_{(d|n)} f(d).$$

In another paper [19], Carlitz proved that for the quotient

$$Q_n = \frac{((p-1)n)!}{(n!)^{p-1}},$$

the highest power, P, of the prime p that divides Q_n is given by

$$P = \begin{cases} 0 & when\ n = \underline{p}_j, \\ aj - j & when\ n = a\underline{p}_j. \end{cases}$$

Carlitz has also used $\underline{z}_n$ in the development of q–Bernoulli numbers and polynomials [6]. He used the notation

$$[x] = \frac{q^x - 1}{q - 1},$$

but as [x] used to be used commonly for the greatest integer function and as Carlitz himself [4] used [k] to mean

$$[k] = x^{p^{nk}} - x,$$

it is felt that $\underline{z}_n$ is less confusing. Moreover, $\underline{z}_n$ has some other notational advantages [27]. If

$$T_n = \sum_{k=1}^{r} T_{n-k},$$

then T_n is the sum of the rising diagonals of the multinomial triangle generated by $\underline{z}_r^n$ [22, 24, 25]. Hoggatt and Bicknell [2] proved that, for the general r-nomial triangle induced by the expansion $\underline{z}_r^n$ (n = 0, 1, 2, 3, ...), by letting the r-nomial triangle be left-justified and by taking sums from the left edge and jumping up p and over 1 entry until out of the triangle that

$$T_n = \sum_{k=0}^{\left\lfloor \frac{1}{2} n(r-1) \right\rfloor} \left\{ \begin{matrix} n-k \\ k \end{matrix} \right\}_r,$$

where

$$\underline{z}_r^n = \sum_{j=0}^{n(r-1)} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_r x^j,$$

and the r-nomial coefficient $\left\{ \begin{matrix} n \\ j \end{matrix} \right\}_r$ is the entry in the n-th row and j-th column of the generalized Pascal triangle [1]. Thus,

$$\left(1 - x\underline{\left(x^p\right)}_r\right)^{-1} = \sum_{n=0}^{\infty} \left(x\underline{\left(x^p\right)}_r\right)^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left\lfloor \frac{n(r-1)}{p+1} \right\rfloor} \left\{ \begin{matrix} n-kp \\ k \end{matrix} \right\}_r \right) x^n$$

which, when p = 1, is a generating function for $\{T_n\}$ with suitable initial values. Here,

$$\underline{\left(x^p\right)}_r = 1 + x^p + x^{2p} + \ldots + x^{p(r-1)}$$

so that the notation is quite versatile.

3 Fontené–Jackson type derivatives

Morgan Ward [38] once posed the problem whether a suitable definition for the generalized Bernoulli numbers could be framed so that a generalized Staudt–Clausen theorem existed within the framework of the Fontené–Jackson calculus (Equation (4.1)). Elsewhere [28], we have shown that it is possible. Here, we focus on two related and relevant differential operators within this calculus in the style of the work of Carlitz with Chak and Shur derivatives [6, 10]. They are defined [34] for notational convenience by

$$D_{zx}x^n = (1-z^n)x^{n-1}, \tag{3.1}$$

and

$$D_{xz} = \underline{z}_n x^{n-1}, \tag{3.2}$$

so that

$$D_{zx}x^n = (1-z^n)x^{n-1} = (1-z)D_{xz}x^n$$

and

$$D_{x1}x^n = nx^{n-1}.$$

Some fundamental properties of D_{xz} follow if we define $D_{zx} = 0$ and

$$D_{xz}ax^n = aD_{xz}x^n$$

in which a is a constant, and for $f(y)$, a function of y,

$$D_{xz}f(y) = D_{yz}f(y)D_{xz}y\,,$$

which, when $z = 1$, reduces to

$$D_x f(y) = D_y f(y) D_x y.$$

Similarly, we can define formally a difference operator, I_{xz}, such that

$$I_{xz}D_{xz} = f(x) \tag{3.3}$$

and

$$D_{xz}f(x) = I_{xz}^{-1}f(x). \tag{3.4}$$

We have, for $n \neq -1$,

$$\begin{aligned} I_{xz}x^n &= \frac{1-z}{1-z^{n+1}}x^{n+1} + C \\ &= \frac{x^{n+1}}{\underline{z}_{n+1}} + C \end{aligned}$$

where C is a constant determined by the initial conditions, and for $n = -1$, we have

$$I_{xz}x^{-1} = L_z(x) + C$$

in which $L_z(x)$ is defined by

$$L_z(1+x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{r+1}}{z^r\,\underline{z}_{r+1}}$$

to accompany $E_z(x)$:

$$E_z(x) = \sum_{n=0}^{\infty}\frac{x^n}{z^r\,\underline{z}_n!}$$

so that

$$D_{xz}L_z(x) = (E_z(y))^{-1}$$

and

$$x = L_z(E(x)).$$

4 Staudt–Clausen Theorem

Carlitz [13, 14, 16] outlined partial solutions to Ward's question (Section 2). A complete solution may be found in [29]. The (von) Staudt–Clausen theorem states that

$$B_{2n} = A_n - \sum_{\substack{p_k \\ (p_k - 1)|2n}} \frac{1}{p_k} \tag{4.1}$$

in which B_{2n} is a Bernoulli number, A_n is an integer, and the p_k are primes such that $(p_k - 1)$ divides $2n$.

We can define generalized Bernoulli numbers in the context of the previous two sections by

$$\frac{t}{E_z(t) - 1} = \sum_{n=0}^{\infty} B_{nz} \frac{t^n}{\underline{z}_n!} \tag{4.2}$$

in which

$$E_z(t) = \sum_{n=0}^{\infty} \frac{t^n}{\underline{z}_n!} \tag{4.3}$$

so that $B_{n1} = B_n$. The B_{nz} are thus generalizations of the ordinary Bernoulli numbers, B_n, for which the Staudt–Clausen theorem can be restated as

$$pB_n \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid n) \\ 0 \pmod{p} & (p-1 \nmid n) \end{cases} \tag{4.4}$$

where p is an odd prime (and hence n is even).

Another generalization of the Staudt–Clausen theorem relevant to our later work is that of Vandiver [37]. He defined generalized Bernoulli numbers of the first order, $b_n(m,k)$, by the umbral equality

$$b_n(m,k) = (mb + k)^n : B_n = b_n(0,1).$$

Vandiver's Staudt–Clausen theorem is that for n even,

$$b_n(m,k) = A_n - \sum_{i=1} \frac{1}{p_i},$$

where the p's are distinct primes relative to m (non-zero) and such that $n \equiv 0 (\operatorname{mod} p_i - 1)$, and A_n is an integer. The generalization of (4.1) is obvious.

5 Generalized Hurwitz series

The Fontené–Jackson type derivatives and integrals of GH–series are also GH–series, since

$$D_{tz} \sum_{n=0}^{\infty} a_n \frac{t^n}{\underline{z}_n!} = \sum_{n=0}^{\infty} a_{n+1} \frac{t^n}{\underline{z}_n!}$$

and

$$I_{xz} \sum_{n=0}^{\infty} a_n \frac{x^n}{\underline{z}_n} \bigg|_0^t = \sum_{n=1}^{\infty} a_{n-1} \frac{t^n}{\underline{z}_n!}$$

For a series without constant term

$$H_1(t) = \sum_{n=1}^{\infty} a_n \frac{t^n}{\underline{z}_n!},$$

it follows from

$$H_1^k(t)/\underline{z}_k! = I_{xz} D_{xz} H_1^k(x)/\underline{z}_k! \Big|_0^t$$

$$= I_{xz} H_1^{k-1}(x) D_{xz} H_1(x)/\underline{z}_{k-1}! \Big|_0^t$$

that

$$H_1^k(t) \equiv 0 (\mathrm{mod}\, \underline{z}_k!), \tag{5.1}$$

because by the statement

$$\sum_{n=0}^{\infty} a_n/\underline{z}_n! \equiv \sum_{n=0}^{\infty} b_n/\underline{z}_n! \; (\mathrm{mod}\, \underline{z}_m)$$

is meant that the system of congruences

$$a_n \equiv b_n \;(\mathrm{mod}\, \underline{z}_m)\; (n = 0,1,2,...)$$

is satisfied. This is equivalent to the assertion that

$$\sum_{n=0}^{\infty} a_n/\underline{z}_n! = \sum_{n=0}^{\infty} b_n/\underline{z}_n! + \underline{z}_m H(t)$$

where $H(t)$ is some generalized Hurwitz series.

6 Concluding comments

There are possible analogous extensions and generalizations of this work, for example, to q–numbers or Gaussian integers such as in Kim [31]. Other possible future research would be to find an analogy to Wilson's theorem with the Fermatian numbers and to find a more comprehensive combinatorial description of them. For instance, Riordan [33] has shown that

$$\underline{q}_{n+1} = 1 + a_{n1}(q)$$

in which $a_{nm}(q)$ is the enumerator of partitions with m parts, none greater then n, such that their Ferrer's graphs include an initial triangle of sides n and m (the graph of partition m, $m - 1$, ..., 2, 1).

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