A comment on a result of Virgolici

Mickey Polasub

9/34 Ramintra 40 Rd.
Nuanchan, Beungkum, Bangkok, Thailand 10230
e-mail: mickey.polasub@gmail.com

Abstract: We provide a short proof of a generalization of a recent result of Virgolici on the diophantine equation $2^x + 1009^y = p^z$.

Keywords: Exponential diophantine equations, Primitive divisors.

AMS Classification: Primary 11D61.

1 Introduction

In a recent paper, Virgolici [3] solved the exponential equation

$$2^x + 1009^y = p^z$$

(1)

for odd primes $p < 1000$, showing that there are no solutions with $y > 0$ (and only solutions with $y = 0$ for the Fermat primes $p = 3, 5, 17$ and $257$).

2 Main result

In this note, we show more generally the following.

Theorem 1 If there exists a solution to equation (1) in integers $x, y$ and $z$ and prime $p$, then either $(x, y, z, p) = (3, 0, 2, 3)$ or $z = 1$.

To prove this, we will assume that we have a solution to (1) with, via Mihailescu’s Theorem, $y > 0$. If $z$ is even, then necessarily $p = 3$, as treated in [3]. Otherwise, we may suppose that $z$ is odd and that, modulo 9, $x$ is even. if $3 | z$, applying Magma’s SIntegralPoints routine (here $S = \{1009\}$) to the elliptic curves $Y^2 = X^3 - 1009^a$ for $0 \leq \alpha \leq 5$, we find no new solutions. If $5 | z$, equation (1) is insoluble modulo

$$11 \cdot 31 \cdot 41 \cdot 61 \cdot 251 \cdot 331 \cdot 401 \cdot 601.$$
We may thus assume that \( z \) is divisible by a prime \( q \geq 7 \). Equation (1) is now a special case of the equation \( a^2 + 1009^y = b^q \), where \( b \) is odd and \( q \geq 7 \) is prime. Appealing to the Primitive Divisor Theorem of Bilu et al [1] in a now-standard way (as in, for example, [2]) and using the fact that \( \mathbb{Q}(\sqrt{-1009}) \) has class number 20, we conclude that there are no solutions to \( a^2 + 1009^y = b^q \) with \( q \geq 7 \). This completes the proof of Theorem 1.

Note that there are many solutions to (1) with \( z = 1 \), likely infinitely many. These are of course just primes of the shape \( 2^x + 1009^y \). There are precisely 20 such primes up to \( 10^{20} \).

It is not too hard to generalize Theorem 1 by replacing \( p \) with an arbitrary integer. This adds the additional solution corresponding to the identity \( 2^9 + 1009 = 39^2 \). Our arguments also work with minor modification if one replaces the prime 1009 with any prime congruent to one of 1, 13 or 19 modulo 24.

**References**

