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Note on φ and ψ functions

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Abstract: In this paper, we improve the lower bound of the paper [3].
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1 Introduction

The paper is a continuation of [1, 2, 3]. In 2011, Atanassov [3] proved: For each natural number $n > 1$, the inequality

$$n^{2n} < \varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}$$

holds (See [4] for φ and ψ). Our aim in this paper is to prove the following theorem. Let

$$\mu(n) = \frac{1}{2}\left(\prod_{p|n}\left(1-\frac{1}{p}\right) + \prod_{p|n}\left(1+\frac{1}{p}\right)\right).$$

Then it is clear that for $n = p_1^{a_1}...p_k^{a_k}$ with distinct primes p_1, ...,p_k and positive integers a_1, ..., a_k, we get

$$\mu(n) = 1 + \sum_{i_1 \neq i_2} \frac{1}{p_{i_1}p_{i_2}} + \sum_{\substack{i_1,\, i_2,\, i_3,\, i_4 \\ distinct}} \frac{1}{p_{i_1}p_{i_2}p_{i_3}p_{i_4}} + ... \tag{1}$$

Theorem 1. *For each natural number $n > 1$, the inequality*

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \geq n^{2n\mu(n)}.$$

holds.

By (1), $\mu(n) > 1$. Then, we have the following result from Theorem 1.

Corollary 1. *For each natural number $n > 1$, we have*

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \geq n^{2n}.$$

2 Proof of Theorem 1

By the definition of φ and ψ, we have

$$\varphi(n)log(\varphi(n)) = (n\log n)\prod_{p|n}\left(1-\frac{1}{p}\right) + n\prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\log\left(1-\frac{1}{p}\right) \tag{2}$$

and

$$\psi(n)log(\psi(n)) = (n\log n)\prod_{p|n}\left(1+\frac{1}{p}\right) + n\prod_{p|n}\left(1+\frac{1}{p}\right)\sum_{p|n}\log\left(1+\frac{1}{p}\right). \tag{3}$$

Next we shall find the value of

$$S = \varphi(n)log(\varphi(n)) + \psi(n)log(\psi(n)).$$

From (2) and (3), we have

$$S = 2\mu n\log n + n\left[\prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\log\left(1-\frac{1}{p}\right) + \prod_{p|n}\left(1+\frac{1}{p}\right)\sum_{p|n}\log\left(1+\frac{1}{p}\right)\right].$$

We have

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \geq \prod_{p|n}\left(1-\frac{1}{p}\right).$$

Hence

$$\sum_{p|n}\ln\left(1+\frac{1}{p}\right) \geq 0 \geq \sum_{p|n}\ln\left(1-\frac{1}{p}\right)$$

and

$$\prod_{p|n}\left(1+\frac{1}{p}\right)\sum_{p|n}\ln\left(1+\frac{1}{p}\right) \geq 0 \geq \prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\ln\left(1-\frac{1}{p}\right).$$

For every prime p

$$\ln\left(1+\frac{1}{p}\right) + \ln\left(1-\frac{1}{p}\right) = \ln\frac{p+1}{p-1} > 0.$$

Hence

$$\sum_{p|n}\ln\left(1+\frac{1}{p}\right) > |\sum_{p|n}\ln\left(1-\frac{1}{p}\right)|.$$

Therefore,

$$\prod_{p|n}\left(1+\frac{1}{p}\right)\sum_{p|n}\ln\left(1+\frac{1}{p}\right) \geq \prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\ln\left(1-\frac{1}{p}\right) \geq 0.$$

From this, we have

$$\varphi(n)log(\varphi(n)) + \psi(n)log(\psi(n)) \geq 2n\mu(n)\log n,$$

This implies that

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \geq n^{2n\mu(n)}.$$

This proves the theorem.

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