Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 1, 25–36

Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials

Mustafa Asci* and Esref Gurel

Department of Mathematics, Science and Arts Faculty
Pamukkale University, Kınıklı Denizli, Turkey
e-mails: mustafa.asci@yahoo.com, esrefgurel@hotmail.com
* Corresponding author

Abstract: In this study we define and study the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials. We give generating function, Binet formula, explicit formula, Q matrix, determinantal representations and partial derivation of these polynomials. By defining these Gaussian polynomials for special cases $GJ_n(1)$ is the Gaussian Jacobsthal numbers, $Gj_n(1)$ is the Gaussian Jacobsthal Lucas numbers defined in [2].
Keywords: Jacobsthal polynomials, Jacobsthal Lucas polynomials, Gaussian Fibonacci numbers.
AMS Classification: 11A07, 11A41, 11A51, 11B50, 11B65, 11B75.

1 Introduction

The complex Fibonacci numbers, Gaussian Fibonacci numbers and their interesting properties are studied by some authors [3–13]. The authors in [1] defined the Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials $GF_n(x, y)$ and $GL_n(x, y)$. They give generating function, Binet formula, explicit formula and partial derivation of these polynomials. Special cases of these bivariate polynomials are Gaussian Fibonacci polynomials $F_n(x, 1)$, Gaussian Lucas polynomials $L_n(x, 1)$, Gaussian Fibonacci numbers $F_n(1, 1)$ and Gaussian Lucas numbers $L_n(1, 1)$ defined in [12]. Also the authors in [2] defined and studied the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. They give generating functions, Binet formulas, explicit formulas and Q matrix of these numbers. They also present explicit combinatorial and determinantal expressions, study negatively subscripted numbers and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers they give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

Horadam [7] defined the Jacobsthal and the Jacobsthal Lucas sequences J_n and j_n by the following recurrence relations

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2$$

where $J_0 = 2$ and $J_1 = 1$ respectively.

Jacobsthal and the Jacobsthal Lucas polynomial sequences $J_n(x)$ and $j_n(x)$ are defined by the following recurrence relations

$$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}\ (x) \text{ for } n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x) \text{ for } n \geq 2$$

where $J_0 = 2x$ and $J_1 = 1$ respectively.

The Gaussian Fibonacci sequence in [12] is $GF_0 = i$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. One can see that

$$GF_n = F_n + iF_{n-1}$$

where F_n is the nth usual Fibonacci number.

The Gaussian Lucas sequence in [12] is defined similar to Gaussian Fibonacci sequence as $GL_0 = 2 - i$, $GL_1 = 1 + 2i$, and $GL_n = GL_{n-1} + GL_{n-2}$ for $n > 1$. Also it can be seen that

$$GL_n = L_n + iL_{n-1}$$

where L_n is the usual nth Lucas number.

The authors [2] defined the Gaussian Jacobsthal and the Gaussian Jacobsthal Lucas sequences GJ_n and Gj_n by the following recurrence relations

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, \quad n \geq 1 \tag{1.1}$$

with initial conditions $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$.

It can be easily seen that $GJ_n = J_n + iJ_{n-1}$, where J_n is the n'th Jacobsthal number.

The Gaussian Jacobsthal Lucas sequences $\{Gj_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$Gj_{n+1} = Gj_n + 2Gj_{n-1} \quad n \geq 1 \tag{1.2}$$

with initial conditions $Gj_0 = 2 - \frac{i}{2}$ and $Gj_1 = 1 + 2i$.

Also $Gj_n = j_n + ij_{n-1}$, where j_n is the nth Jacobsthal Lucas number.

In this study we define and study the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials. Special cases of these polynomials are Gaussian Jacobsthal numbers $GJ_n(1)$ and Gaussian Jacobsthal Lucas numbers $GL_n(1)$ defined in [2]. We give generating functions, Binet formulas, explicit formulas and Q matrix of these polynomials. We also present explicit combinatorial and determinantal expressions, and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas polynomials we give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials.

2 The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials

Definition 1. *The Gaussian Jacobsthal polynomials $\{GJ_n(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x), \quad n \geq 1 \tag{2.1}$$

with initial conditions $GJ_0(x) = \frac{i}{2}$ and $GJ_1(x) = 1$.

It can be easily seen that

$$GJ_n(x) = J_n(x) + ixJ_{n-1}(x)$$

where $J_n(x)$ is the n'th Jacobsthal polynomial.

Definition 2. *The Gaussian Jacobsthal Lucas polynomials $\{Gj_n(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$Gj_{n+1}(x) = Gj_n(x) + 2xGj_{n-1}(x) \quad n \geq 1 \tag{2.2}$$

with initial conditions $Gj_0(x) = 2 - \frac{i}{2}$and $Gj_1(x) = 1 + 2ix$.

Also $Gj_n(x) = j_n(x) + ixj_{n-1}(x)$, where $j_n(x)$ is the nth Jacobsthal Lucas polynomial.

We observe that $GJ_n(1) = GJ_n$ and $Gj_n(1) = Gj_n$ where GJ_n and Gj_n are the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers defined in 1.1 and 1.2 recursively .

For later use the first few terms of the sequences are shown in the following table

n	$GJ_n(x)$	$Gj_n(x)$
0	$\frac{i}{2}$	$2 - \frac{i}{2}$
1	1	$1 + 2xi$
2	$1 + xi$	$4x + 1 + xi$
3	$2x + 1 + xi$	$6x + 1 + (4x + 1)xi$
4	$4x + 1 + (2x + 1)xi$	$8x^2 + 8x + 1 + (6x + 1)xi$
5	$4x^2 + 6x + 1 + (4x + 1)xi$	$20x^2 + 10x + 1 + (8x^2 + 8x + 1)xi$
6	$12x^2 + 8x + 1 + (4x^2 + 6x + 1)xi$	$16x^3 + 36x^2 + 12x + 1 + (20x^2 + 10x + 1)xi$
$\vdots$	$\vdots$	$\vdots$

2.1 Some properties

Theorem 1. *The generating function for Gaussian Jacobsthal polynomials is*

$$g(t,x)=\sum_{n=0}^{\infty} GJ_n(x)t^n=\frac{2t+i(1-t)}{2-2t-4xt^2}$$

and for Gaussian Jacobsthal Lucas polynomials is

$$h(t,x)=\sum_{n=0}^{\infty} Gj_n(x)t^n=\frac{4-2t+i(t-1+4xt)}{2-2t-4xt^2}.$$

Proof. Let $g\,(t,x)$ be the generating function of Gaussian Jacobsthal polynomial sequence $GJ_n(x)$, then

$$\begin{aligned}
g(t,x)-tg(t,x)-2xt^2g(t,x) &= GJ_0(x)+GJ_1(x)t-GJ_0(x)t \\
&\quad +\sum_{n=2}^{\infty} t^n\left[GJ_n(x)-GJ_{n-1}(x)-2xGJ_{n-2}(x)\right] \\
&= \frac{i}{2}+\left(1-\frac{i}{2}\right)t \\
&= \frac{2t+i\,(1-t)}{2}
\end{aligned}$$

by taking $g(t,x)$ parenthesis we get

$$g(t,x)=\frac{2t+i(1-t)}{2-2t-4xt^2}.$$

□

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation

$$t^2-t-2x=0$$

of the recurrence relation (2.1). Then

$$\alpha(x)=\frac{1+\sqrt{8x+1}}{2},\quad \beta(x)=\frac{1-\sqrt{8x+1}}{2}.$$

Note that $\alpha(x)+\beta(x)=1$ and $\alpha(x)\beta(x)=-2x$. Now we can give the Binet formula for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials.

Theorem 2. *For* $n\geq 0$

$$GJ_n(x)=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}+ix\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}$$

and

$$Gj_n(x) = \alpha^n(x) + \beta^n(x) + ix\left(\alpha^{n-1}(x) + \beta^{n-1}(x)\right).$$

Proof. Theorem can be proved by mathematical induction on n $\square$

Theorem 3. *The explicit formula of Gaussian Jacobsthal Polynomials is*

$$GJ_n(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} (2x)^k + i \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \binom{n-k-2}{k} 2^k x^{k+1}.$$

Theorem 4. *The explicit formula of Gaussian Jacobsthal Lucas Polynomials is*

$$Gj_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} (2x)^k + i \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{n-1}{n-k-1} \binom{n-k-1}{k} 2^k x^{k+1}.$$

Theorem 5. *Let $D_n(x)$ denote the $n \times n$ tridiagonal matrix as*

$$D_n(x) = \begin{bmatrix} 1 & i & 0 & \cdots & 0 \\ -x & 1 & 2x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2x \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, n \geq 1$$

and let $D_0(x) = \frac{i}{2}$. Then

$$\det D_n(x) = GJ_n(x), n \geq 1.$$

Proof. By induction on n we can prove the theorem. For $n = 1$ and $n = 2$

$$\begin{aligned} \det D_1(x) &= 1 = GJ_1(x) \\ \det D_2(x) &= 1 + xi = GJ_2(x) \end{aligned}$$

Assume that the theorem is true for $n-1$ and $n-2$

$$\begin{aligned} \det D_{n-1}(x) &= GJ_{n-1}(x) \\ \det D_{n-2}(x) &= GJ_{n-2}(x) \end{aligned}$$

Then

$$\begin{aligned} \det D_n(x) &= \det D_{n-1}(x) + 2x \det D_{n-2}(x) \\ &= GJ_{n-1}(x) + 2xGJ_{n-2}(x) \\ &= GJ_n(x) \end{aligned}$$

$\square$

Theorem 6. *Let $H_n(x)$ denote the $n \times n$ tridiagonal matrix defined as*

$$H_n(x) = \begin{bmatrix} 2-\frac{i}{2} & \frac{x}{2}-1 & 0 & \cdots & 0 \\ 1 & ix & 2x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2x \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad n \geq 1.$$

Then

$$\det H_n(x) = Gj_{n-1}(x), n \geq 0.$$

Proof. By induction on n we can prove the theorem. For $n = 1$and $n = 2$

$$\begin{aligned} \det D_1(x) &= 2 - \frac{i}{2} = Gj_0(x) \\ \det D_2(x) &= 1 + 2xi = Gj_1(x) \end{aligned}$$

Assume that the theorem is true for $n-1$ and $n-2$

$$\begin{aligned} \det D_{n-1}(x) &= Gj_{n-2}(x) \\ \det D_{n-2}(x) &= GjJ_{n-3}(x) \end{aligned}$$

Then

$$\begin{aligned} \det D_n(x) &= \det D_{n-1}(x) + 2x \det D_{n-2}(x) \\ &= Gj_{n-2}(x) + 2xGj_{n-3}(x) \\ &= Gj_{n-1}(x) \end{aligned}$$

□

Now we introduce the matrices $Q(x)$ and P that plays the role of the Q-matrix of Fibonacci numbers. Let $Q(x)$ and P denote the 2×2 matrices defined as

$$Q(x) = \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}, \ P = \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \text{ and } R = \begin{bmatrix} 4x+1+xi & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix}$$

Then we can give the following theorems:

Theorem 7. *Let $n \geq 1$. Then*

$$Q^n(x)P = \begin{bmatrix} GJ_{n+2}(x) & GJ_{n+1}(x) \\ GJ_{n+1}(x) & GJ_n(x) \end{bmatrix}$$

where $GJ_n(x)$ is the nth Gaussian Jacobsthal Polynomial.

Proof. We can prove the theorem by induction on n For $n = 1$

$$\begin{aligned}
\begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix} &= \begin{bmatrix} 2x+1+ix & 1+ix \\ 1+ix & 1 \end{bmatrix} \\
&= \begin{bmatrix} GJ_3(x) & GJ_2(x) \\ GJ_2(x) & GJ_1(x) \end{bmatrix}
\end{aligned}$$

Assume that the theorem holds for$n = k$, that is

$$\begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix} = \begin{bmatrix} GJ_{k+2}(x) & GJ_{k+1}(x) \\ GJ_{k+1}(x) & GJ_k(x) \end{bmatrix}$$

Then for $n = k+1$ we have

$$\begin{aligned}
\begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix} &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GJ_{k+2}(x) & GJ_{k+1}(x) \\ GJ_{k+1}(x) & GJ_k(x) \end{bmatrix} \\
&= \begin{bmatrix} GJ_{k+3}(x) & GJ_{k+2}(x) \\ GJ_{k+2}(x) & GJ_{k+1}(x) \end{bmatrix}
\end{aligned}$$

□

Theorem 8. *Let $n \geq 1$. Then*

$$Q^n(x) R = \begin{bmatrix} Gj_{n+2}(x) & Gj_{n+1}(x) \\ Gj_{n+1}(x) & Gj_n(x) \end{bmatrix}$$

where $Gj_n(x)$ is the nth Gaussian Jacobsthal-Lucas Polynomial.

Proof. We can prove the theorem by induction on n For $n = 1$

$$\begin{aligned}
Q(x) R &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4x+1+xi & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix} \\
&= \begin{bmatrix} 6x+1+ix(4x+1) & 4x+1+ix \\ 4x+1+ix & 1+2ix \end{bmatrix} \\
&= \begin{bmatrix} Gj_3(x) & Gj_2(x) \\ Gj_2(x) & Gj_1(x) \end{bmatrix}
\end{aligned}$$

Assume that the theorem holds for $n = k$, that is

$$\begin{aligned} Q^k(x)\,R &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 4x+1+xi & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} Gj_{k+2}(x) & Gj_{k+1}(x) \\ Gj_{k+1}(x) & Gj_k(x) \end{bmatrix} \end{aligned}$$

Then for $n = k+1$ we have

$$\begin{aligned} Q^{k+1}(x)\,R &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 4x+1+xi & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}^{k} \begin{bmatrix} 4x+1+xi & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Gj_{k+2}(x) & Gj_{k+1}(x) \\ Gj_{k+1}(x) & Gj_k(x) \end{bmatrix} \\ &= \begin{bmatrix} Gj_{k+3}(x) & Gj_{k+2}(x) \\ Gj_{k+2}(x) & Gj_{k+1}(x) \end{bmatrix} \end{aligned}$$

□

Theorem 9. *(Cassini Identity) For* $n \geq 1$

$$GJ_{n-1}(x)GJ_{n+1}(x) - GJ_n^2(x) = (-1)^n\, 2^{n-2}x^{n-1}\,(2+x-i)$$

Proof. We can prove the theorem by matrices method

First of all we determine the determinants of the matrices

$$\det Q^{n-1}(x) = \begin{vmatrix} 1 & 2x \\ 1 & 0 \end{vmatrix}^{n-1} = (-2x)^{n-1}$$

$$\det P = \begin{vmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{vmatrix} = -\frac{1}{2}\,(2+x-i)$$

by the previous theorem

$$Q^{n-1}(x)\,P = \begin{bmatrix} GJ_{n+1}(x) & GJ_n(x) \\ GJ_n(x) & GJ_{n-1}(x) \end{bmatrix}$$

we get the determinants of the matrices

$$\begin{aligned} GJ_{n+1}(x)GJ_{n-1}(x) - GJ_n^2(x) &= \det\left(Q^{n-1}(x)\,P\right) \\ &= \det Q^{n-1}(x)\det P \\ &= (-2x)^{n-1}\frac{-1}{2}(2+x-i) \\ &= (-1)^n\,2^{n-2}x^{n-1}(2+x-i) \end{aligned}$$

As a result

$$GJ_{n+1}(x)GJ_{n-1}(x) - GJ_n^2(x) = (-1)^n\,2^{n-2}x^{n-1}(2+x-i)$$

□

Theorem 10. *For* $n \geq 1$

$$Gj_{n-1}(x)Gj_{n+1}(x) - Gj_n^2(x) = (8x+1)(x+2-i)(-1)^{n-1}\,2^{n-2}x^{n-1}.$$

Proof. We can prove the theorem by matrices method by the previous theorem

$$Q^{n-1}(x)\,R = \begin{bmatrix} Gj_{n+1}(x) & Gj_n(x) \\ Gj_n(x) & Gj_{n-1}(x) \end{bmatrix}$$

we get the determinants of the matrices

$$\begin{aligned} GJ_{n+1}(x)GJ_{n-1}(x) - GJ_n^2(x) &= \det\left(Q^{n-1}(x)\,R\right) \\ &= \det Q^{n-1}(x)\det R \\ &= (-2x)^{n-1}\frac{1}{2}(8x+1)(x+2-i) \\ &= (-1)^{n-1}\,2^{n-2}x^{n-1}(8x+1)(2+x-i) \end{aligned}$$

As a result

$$Gj_{n-1}(x)Gj_{n+1}(x) - Gj_n^2(x) = (8x+1)(x+2-i)(-1)^{n-1}\,2^{n-2}x^{n-1}.$$

□

Theorem 11. *For* $n \geq 1$

$$Gj_n^2(x) - (1+8x)\,GJ_n^2(x) = (x+2-i)(-1)^n\,2^{n+1}x^n.$$

The theorem can be proved by induction on n, thus we omit the proof.

Corollary 1. *[2] (Cassini Identity) If* $x = 1$ *then*

$$GJ_{n-1}GJ_{n+1} - GJ_n^2 = (3-i)(-1)^n\,2^{n-2}$$

Corollary 2. *[2] If* $x = 1$ *then*

$$Gj_{n-1}Gj_{n+1} - Gj_n^2 = 9\,(3-i)\,(-1)^{n-1}\,2^{n-2}$$

Theorem 12. *For* $n \geq 1$

$$Gj_n(x) = GJ_{n+1}(x) + 2xGJ_{n-1}(x).$$

The theorem can be proved by induction on n, thus we omit the proof.

Corollary 3. *[2] If* $x = 1$ *then*

$$Gj_n = GJ_{n+1} + 2GJ_{n-1}$$

Theorem 13. *For* $n \geq 1$

$$GJ_n(x) = \frac{Gj_{n+1}(x) + 2xGj_{n-1}(x)}{1+8x}.$$

The theorem can be proved by induction on n, thus we omit the proof.

Corollary 4. *[2] If* $x = 1$ *then*

$$GJ_n = \frac{Gj_{n+1} + 2Gj_{n-1}}{9}$$

Theorem 14. *The sums of the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials are given as:*

(i) $\sum\limits_{k=0}^{n} GJ_k(x) = \frac{1}{2x}\left[GJ_{n+2}(x) - 1\right]$

(ii) $\sum\limits_{k=0}^{n} Gj_k(x) = \frac{1}{2x}\left[Gj_{n+2}(x) - (1+2xi)\right]$

Proof. For $n \geq 1$ we have $GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x)$

$$GJ_{n-1}(x) = \frac{1}{2x}\left(GJ_{n+1}(x) - GJ_n(x)\right)$$

From this equation

$$\begin{aligned}
GJ_0(x) &= \frac{1}{2x}\left(GJ_2(x) - GJ_1(x)\right) \\
GJ_1(x) &= \frac{1}{2x}\left(GJ_3(x) - GJ_2(x)\right) \\
GJ_2(x) &= \frac{1}{2x}\left(GJ_4(x) - GJ_3(x)\right) \\
&\vdots \\
GJ_{n-1}(x) &= \frac{1}{2x}\left(GJ_{n+1}(x) - GJ_n(x)\right) \\
GJ_n(x) &= \frac{1}{2x}\left(GJ_{n+2}(x) - GJ_{n+1}(x)\right)
\end{aligned}$$

By adding both sides of the equation we get

$$\begin{aligned}\sum_{k=0}^{n} GJ_k(x) &= \frac{1}{2x}\left(GJ_{n+2}(x) - GJ_1(x)\right) \\ &= \frac{1}{2x}\left(GJ_{n+2}(x) - 1\right)\end{aligned}$$

this completes the proof $\square$

Theorem 15. *For $n \geq 1$*

$$GJ_n\left(x\right) Gj_n\left(x\right) = GJ_{2n}\left(x\right) + ixJ_{2n-1}\left(x\right) - x^2 J_{2n-2}\left(x\right)$$

whereJ_n is the nth Jacobsthal polynomial.

Proof. By Binet formulas of $GJ_n\left(x\right)$ and $Gj_n\left(x\right)$

$$\begin{aligned}GJ_n(x)Gj_n(x) &= \left[\frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + ix\frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)}\right] \\ &\quad \times \left[\alpha^n(x) + \beta^n(x) + ix\left(\alpha^{n-1}(x) + \beta^{n-1}(x)\right)\right] \\ &= \frac{\alpha^{2n}(x) - \beta^{2n}(x)}{\alpha(x) - \beta(x)} + ix\frac{\alpha^{2n-1}(x) - \beta^{2n-1}(x)}{\alpha(x) - \beta(x)} \\ &\quad + ix\frac{\alpha^{2n-1}(x) - \beta^{2n-1}(x)}{\alpha(x) - \beta(x)} - x^2\frac{\alpha^{2n-2}(x) - \beta^{2n-2}(x)}{\alpha(x) - \beta(x)} \\ &= GJ_{2n}\left(x\right) + ixJ_{2n-1} - x^2 J_{2n-2}\end{aligned}$$

This completes the proof $\square$

Corollary 5. *[2] If $x = 1$ then*

$$GJ_n Gj_n = GJ_{2n} + iJ_{2n-1} - J_{2n-2}.$$

Theorem 16. *For the partial derivatives we have*

$$\frac{\partial Gj_n(x)}{\partial x} = 2nGJ_{n-1}\left(x\right) + iJ_n\left(x\right)$$

where $J_n(x)$ is the nth Jacobsthal polynomial.

The theorem can be proved from the partial derivation of the explicit formula of Gaussian Jacobsthal Lucas polynomials.

References

[1] Asci, M., E. Gurel. Bivariate Gaussian Fibonacci and Lucas Polynomials (to appear in Ars. Comb).

[2] Asci M., Gurel E. "Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers" (to appear in Ars. Comb).

[3] Berzsenyi, G. Gaussian Fibonacci numbers. *Fibonacci Quarterly*, Vol. 15, 1977, No. 3, 233–236.

[4] Good, I. J. Complex Fibonacci and Lucas numbers, continued fractions, and the square root of the golden ratio. *Fibonacci Quarterly*, Vol. 31, 1993, No. 1, 7–20.

[5] Berzsenyi, G. Gaussian Fibonacci Numbers. *Fibonacci Quarterly*, Vol. 15, 1977, 233–236.

[6] Harman, C. J. Complex Fibonacci numbers. *Fibonacci Quarterly*, Vol. 19, 1981, No. 1, 82–86.

[7] Horadam, A. F. Jacobsthal Representation Numbers, *Fibonacci Quarterly*, Vol. 34, 1996, 40–54.

[8] Horadam, A. F. Generating Functions for Powers of a Certain Generalized Sequence of Numbers.*Duke Math. J.*, Vol. 32, 1965, 437–46.

[9] Horadam, A. F. A Generalized Fibonacci Sequence, *American Math. Monthly*, Vol. 68, 1961, 455–459.

[10] Horadam, A. F. Complex Fibonacci Numbers and Fibonacci Quaternions. *American Math. Monthly*, Vol. 70, 1963, 289–291.

[11] Horadam, A. F., E. M. Horadam. Roots of recurrence-generated polynomials. *Fibonacci Quarterly*, Vol. 20, 1982, No. 3, 219–226.

[12] Jordan, J. H. Gaussian Fibonacci and Lucas numbers. *Fibonacci Quarterly*, Vol. 3, 1965, 315–318.

[13] Pethe, S., A. F. Horadam. Generalized Gaussian Fibonacci numbers. *Bull. Austral. Math. Soc.*, Vol. 33, 1986, No. 1, 37–48.