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Abstract: We determine the best positive constants $a$ and $b$ such that

$$\left(\frac{\cos x + 2}{3}\right)^a < \frac{\sin x}{x} < \left(\frac{\cos x + 2}{3}\right)^b.$$  

Similar sharp inequalities are also considered.

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1 Introduction

In paper [4] the author has determined the best positive constants $p$ and $q$ such that

$$\left(\frac{\sinh x}{x}\right)^p < \frac{x}{\sin x} < \left(\frac{\sinh x}{x}\right)^q,$$  \hspace{0.5cm} (1.1)

where $x \in (0, \pi/2)$. In fact one has $p = 1$ and $q \approx 1.18$. Similar results have been obtained in paper [3]:

The best constants $r, s > 0$ such that

$$\frac{1}{(\cosh x)^r} < \frac{\sin x}{x} < \frac{1}{(\cosh x)^s}, \hspace{0.5cm} x \in \left(0, \frac{\pi}{2}\right)$$  \hspace{0.5cm} (1.2)

are $r \approx 0.49 \ldots$, $s = \frac{1}{3}$.  

The best constants $u, v > 0$ such that

$$\left(\frac{\sinh x}{x}\right)^u < \frac{2}{\cos x + 1} < \left(\frac{\sinh x}{x}\right)^v, \hspace{0.5cm} x \in \left(0, \frac{\pi}{2}\right)$$  \hspace{0.5cm} (1.3)

are $u = 3/2, v \approx 1.81 \ldots$.  

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The famous Cusa-Huygens inequality (see e.g. [2]) states that for any \( x \in (0, \pi/2) \) one has
\[
\frac{\sin x}{x} < \frac{\cos x + 2}{3}.
\tag{1.4}
\]

As it is well-known that (see e.g. [2])
\[
\frac{\sin x}{x} > \frac{\cos x + 1}{2},
\tag{1.5}
\]
and as an immediate computation gives
\[
\frac{\cos x + 1}{2} > \left( \frac{\cos x + 2}{3} \right)^2
\]
(equivalent with \((\cos x - 1)(2\cos x + 1) < 0\)), clearly one arises the question on the constants \( a, b > 0 \) such that
\[
\left( \frac{\cos x + 2}{3} \right)^a < \frac{\sin x}{x} < \left( \frac{\cos x + 2}{3} \right)^b.
\tag{1.6}
\]

Similarly, as it is shown in [2], one has
\[
\frac{\sin x}{x} > \left( \frac{\cos x + 1}{2} \right)^{2/3},
\]
by (1.5) we can study the constants \( c \) and \( d > 0 \) such that
\[
\left( \frac{\cos x + 1}{2} \right)^c < \frac{\sin x}{x} < \left( \frac{\cos x + 1}{2} \right)^d,
\tag{1.7}
\]
where, as in the case of (1.6), \( x \in (0, \pi/2) \).

The hyperbolic variants of these inequalities may be studied, too. In what follows, we shall always assume that \( x \in (0, \pi/2) \).

\section{Main results}

First, by using the method of [4] we shall prove the following:

\textbf{Theorem 2.1.} The best positive constants \( a \) and \( b \) in inequality (1.6) are \( a = (\ln \pi/2)/(\ln 3/2) \approx 1.113, \ldots \) and \( b = 1 \).

\textbf{Proof.} We shall use the following auxiliary results:

\textbf{Lemma 2.1.} One has, for any \( x \in (0, \pi/2) \), the inequalities
\[
\ln \frac{x}{\sin x} < \frac{\sin x - x \cos x}{2 \sin x}
\tag{2.1}
\]
and
\[
\ln \frac{3}{2 + \cos x} > \frac{x \sin x}{2(2 + \cos x)}.
\tag{2.2}
\]

\textbf{Proof.} Inequality (2.1) is proved in [4] (see Lemma 2.2). For the proof of (2.2) consider the application
\[
a(x) = \ln \frac{3}{2 + \cos x} - \frac{x \sin x}{2(2 + \cos x)},
\]
\( x \in \left[0, \frac{\pi}{2}\right] \).
For the derivative of this function one can deduce, by an elementary computation:
\[
2(2 + \cos x)^2 a'(x) = 2 \sin x + \sin x \cos x - 2x \cos x - x = b(x).
\]

Now \(b'(x) = 2 \sin x(x - \sin x) > 0\), and as \(b(0) = 0\), we get \(b(x) \geq b(0) = 0\) for \(x \geq 0\). This in turn implies \(a'(x) \geq 0\) for \(x \geq 0\) and as \(a(0) = 0\), we get \(a(x) > 0\) for \(x > 0\) and \(x < \pi/2\). This proves relation (2.2) of Lemma 2.1.

**Proof of Theorem 2.1.** Let us introduce the application
\[
h(x) = \frac{\ln(x/\sin x)}{\ln(3/(2 + \cos x)), \ x \in (0, \pi/2)}
\]
and
\[
f(x) = \ln(x/\sin x), \ g(x) = \ln(3/(2 + \cos x)).
\]

One gets easily
\[
g^2(x)h'(x) = \frac{\sin x - x \cos x}{x \sin x} \ln \frac{3}{2 + \cos x} - \frac{\sin x}{2 + \cos x} \ln \frac{x}{\sin x}, \ (2.3)
\]

By inequality (2.1) one can write
\[
g^2(x)h'(x) > \frac{\sin x - x \cos x}{\sin x} \left[ \frac{1}{x} \ln \frac{3}{2 + \cos x} - \frac{\sin x}{2(2 + \cos x)} \right],
\]
so by (2.2), the parenthesis being strictly positive, we get by (2.3)

\[
h'(x) > 0.
\]

Thus \(h(x)\) is a strictly increasing function. This implies
\[
\lim_{x \to 0} h(x) = 1 < h(x) < h \left( \frac{\pi}{2} \right) = \frac{\ln \pi/2}{\ln 3/2},
\]
so we get the best constants in (1.6), \(a = \frac{\ln \pi/2}{\ln 3/2} \approx 1, 113\ldots\) and \(b = 1\). This proves Theorem 2.1.

In what follows, we shall prove by another method the following result:

**Theorem 2.2.** The best positive constants \(c\) and \(d\) in inequality (1.7) are \(c = \frac{2}{3}\) and
\[
d = \frac{\ln(\pi/2)}{\ln 2} \approx 0.651\ldots.
\]

**Proof.** The following variant of L’Hôpital’s rule, known also as the “monotone form of L’Hôpital’s rule” will be applied (see [1], p. 106):

**Lemma 2.2.** For \(a < b\), let \(f, g\) be continuous on \([a, b]\), differentiable on \((a, b)\) and \(g'\) never vanish on \((a, b)\). If \(f'/g'\) is (strictly) increasing (decreasing) on \((a, b)\), then so are \(f(x) - f(a)\) and \(g(x) - g(a)\), and \(f(x) - f(b)\) and \(g(x) - g(b)\).
Let \( f(x) = \ln \frac{2}{\cos x + 1} \) and \( g(x) = \ln \frac{x}{\sin x} \), where \([a, b] = [0, \pi/2]\). Then,

\[
\frac{f'(x)}{g'(x)} = \frac{x \sin^2 x}{(\sin x - x \cos x)(\cos x + 1)} = \frac{2x \sin^2 \frac{x}{2}}{\sin x - x \cos x} = \frac{f_1(x)}{g_1(x)}.
\]

One has

\[
\frac{f_1'(x)}{g_1'(x)} = 2 \sin \frac{x}{2} \left( \sin \frac{x}{2} + x \cos \frac{x}{2} \right) = 1 + \frac{\tan \frac{x}{2}}{x}.
\]

As for \( k(x) = \tan \frac{x}{2} \), one has \( k'(x) = \frac{x - \sin x}{2 \cos^2 \frac{x}{2}} > 0 \), the function \( k(x) \) is strictly increasing. As \( f_1(0) = g_1(0) = 0 \), \( f_1'(0) = g_1'(0) = 0 \), \( f_1(x) = g_1(x) \) will be strictly increasing. This in turn implies the same for the function \( \frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} \). As (1.7) may be written as \( \frac{1}{c} < \frac{h(x)}{d} < 1 \), and as \( h(x) \) is strictly increasing, we get

\[
\frac{1}{c} = \lim_{x \to 0} h(x) = 3, \quad \frac{1}{d} = h\left(\frac{\pi}{2}\right) = \frac{\ln 2}{\ln(\pi/2)}.
\]

This proves Theorem 2.2.

There exist also hyperbolic variants to these theorems. We prove one of these theorems, namely:

**Theorem 2.3.** The best positive constants \( m \) and \( n \) such that

\[
\left( \cosh \frac{x + 1}{2} \right)^m < \frac{\sinh x}{x} < \left( \cosh \frac{x + 1}{2} \right)^n, \quad x > 0
\]

are \( m = \frac{2}{3} \) and \( n = 1 \).

**Proof.** As in the proof of Theorem 2.2, let

\[
h(x) = \ln \left( \frac{\sinh x}{x} \right) / \ln \left( \cosh \frac{x + 1}{2} \right) = \frac{f(x)}{g(x)}, \quad x \in (0, +\infty).
\]

Then, by elementary computations we get

\[
\frac{f'(x)}{g'(x)} = \frac{x \cosh x - \sinh x}{2x \sinh^2 \frac{x}{2}} = \frac{f_1(x)}{g_1(x)}.
\]

One gets

\[
\frac{f_1'(x)}{g_1'(x)} = \frac{x \sinh x}{2 \sinh \frac{x}{2} \left( \sinh \frac{x}{2} + x \cosh \frac{x}{2} \right)} = 1 + \frac{1}{\tan \frac{x}{2}}.
\]

Put \( l(x) = \frac{\tanh \frac{x}{2}}{x} \). As \( l'(x) = \frac{x - \sinh x}{2 \cosh^2 \frac{x}{2}} < 0 \), \( l(x) \) is strictly decreasing for any \( x > 0 \), so \( \frac{f_1'(x)}{g_1'(x)} \) is strictly increasing. As \( f_1'(0) = g_1'(0) = 0 \) and \( f(0) = g(0) = 0 \), \( \frac{f(x)}{g(x)} = h(x) \) will
be strictly increasing on \((0, b)\) for any \(b > 0\). Thus, we get from Lemma 2.2 that \(h(x)\) is strictly increasing for any \(x \in (0, b)\); thus
\[
\lim_{x \to 0} h(x) = \frac{2}{3} < h(x) < h(b) = \ln \frac{\sinh b}{b} / \ln \left( \frac{\cosh b + 1}{2} \right).
\]
As \(\lim_{b \to +\infty} h(b) = 1\), we get \(m = \frac{2}{3}\) and \(n = 1\), so Theorem 2.3 follows.

References


Notes added in proof

This paper was written in 2010, and sent to the journal September 19, 2011.


As one can see, our method is based on the earlier paper [4], while the above work uses completely different (and more complicated) arguments. As this paper appeared 7 December 2011, clearly the result has been sent to journals about the same times independently.