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Short remark on Möbius function

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Abstract: Some representations of Möbius function μ are introduced, and illustrated with an example.
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The well known Möbius function is defined for the natural number $n \geq 2$, with canonical representation

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

(where k, α_1, ... $\alpha_k \geq 1$ – natural numbers and $p_1, ..., p_k$ – different prime numbers), by (e.g, [1, 2]):

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is not a square-free} \\ (-1)^k, & \text{if } n = p_1 \ldots p_k \end{cases}$$

and $\mu(1) = 1$.

Some representations of this function are given below.

Let for the above canonical form of n

$$\underline{set}(n) = \{p_1, ..., p_k\},$$

$$\underline{mult}(n) = \prod_{i=1}^{k} p_i,$$

$$\omega(n) = k,$$

$$\Omega(n) = \sum_{i=1}^{k} \alpha_i,$$

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1),$$

$$sum_m(n) = \sum_{i=1}^{k} p_i^m, \text{ where } m \geq 1 \text{ is a natural number},$$

$$\zeta(n) = \sum_{i=1}^{k} \alpha_i p_i,$$

$$\overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases},$$

where x is a real number.

The following assertion is valid.

Theorem. For every natural number $n \geq 2$

$$\begin{aligned}
&\text{(a) } \mu(n) = (-1)^{\omega(n)} \left[\frac{2^{\omega(n)}}{\tau(n)}\right] \\
&\text{(b)} \qquad = (-1)^{\omega(n)} \overline{\text{sg}}(2^{\omega(n)} - \tau(n)) \\
&\text{(c)} \qquad = (-1)^{\omega(n)} \left[\frac{\underline{mult}(n)}{n}\right] \\
&\text{(d)} \qquad = (-1)^{\omega(n)} \overline{\text{sg}}(\underline{mult}(n) - n) \\
&\text{(e)} \qquad = (-1)^{\omega(n)} \left[\frac{\omega(n)}{\Omega(n)}\right] \\
&\text{(f)} \qquad = (-1)^{\omega(n)} \overline{\text{sg}}(\omega(n) - \Omega(n)) \\
&\text{(g)} \qquad = (-1)^{\omega(n)} \left[\frac{sum_1(n)}{\zeta(n)}\right].
\end{aligned}$$

Proof. Let the natural number n be square free. Therefore, for all i $(1 \leq i \leq k)$, $\alpha_i = 1$ and hence

$$2^{\omega(n)} = \tau(n),$$

$$\underline{mult}(n) = n,$$

$$\omega(n) = \Omega(n),$$

$$sum_1(n) = \zeta(n).$$

Therefore,

$$\left[\frac{2^{\omega(n)}}{\tau(n)}\right] = \overline{\text{sg}}(2^{\omega(n)} - \tau(n)) = \left[\frac{\underline{mult}(n)}{n}\right]$$

$$= \overline{\text{sg}}(\underline{mult}(n) - n) = \left[\frac{\omega(n)}{\Omega(n)}\right] = \overline{\text{sg}}(\omega(n) - \Omega(n)) = \left[\frac{sum_1(n)}{\zeta(n)}\right] = 1.$$

If n has odd number of prime divisors, $(-1)^{\omega(n)} = -1$ and if n has even number of prime divisors, $(-1)^{\omega(n)} = 1$. Therefore, each of the right sides of the seven formulas is a representation of μ.

Let the natural number n be not square free. Therefore, at least for one i $(1 \le i \le k)$, $\alpha_i \ge 2$ and hence

$$2^{\omega(n)} < \tau(n),$$

$$\underline{mult}(n) < n,$$

$$\omega(n) < \Omega(n),$$

$$sum_1(n) < \zeta(n).$$

Therefore,

$$\left[\frac{2^{\omega(n)}}{\tau(n)}\right] = \overline{\mathrm{sg}}(2^{\omega(n)} - \tau(n)) = \left[\frac{\underline{mult}(n)}{n}\right]$$

$$= \overline{\mathrm{sg}}(\underline{mult}(n) - n) = \left[\frac{\omega(n)}{\Omega(n)}\right] = \overline{\mathrm{sg}}(\omega(n) - \Omega(n)) = \left[\frac{sum_1(n)}{\zeta(n)}\right] = 0$$

and obviously, in all these cases $\mu(n) = 0$. Therefore, again each one of the right sides of the six formulas is a representation of μ.

It is valid the following

Lemma. For every natural number $k \ge 1$:

$$(-1)^k = 4\left[\frac{k}{2}\right] - 2k + 1.$$

Proof. Let $k = 2m + 1$. Then,

$$(-1)^k = (-1)^{2m+1} = -1 = 4m - 2(2m + 1) + 1 = 4\left[\frac{k}{2}\right] - 2k + 1.$$

Let $k = 2m$. Then,

$$(-1)^k = (-1)^{2m} = 1 = 4m - 2.(2m) + 1 = 4\left[\frac{k}{2}\right] - 2k + 1.$$

Therefore, formulas (a)–(g) from the above Theorem can be written in the forms

$$\text{(a) } \mu(n) = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) . \left[\frac{2^{\omega(n)}}{\tau(n)}\right]$$

$$\text{(b) } \quad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) .\overline{\mathrm{sg}}(2^{\omega(n)} - \tau(n))$$

$$\text{(c) } \quad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) . \left[\frac{\underline{mult}(n)}{n}\right]$$

$$\text{(d) } \quad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) .\overline{\mathrm{sg}}(\underline{mult}(n) - n)$$

$$\text{(e) } \quad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) . \left[\frac{\omega(n)}{\Omega(n)}\right]$$

$$\text{(f) } \quad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) .\overline{\mathrm{sg}}(\omega(n) - \Omega(n)).$$

$$\text{(g)} \qquad = \left(4\left[\frac{\omega(n)}{2}\right] - 2n + 1\right) . \left[\frac{sum_1(n)}{\zeta(n)}\right].$$

It is well known that for every two real numbers x and y, $|x.y| = |x|.|y|$. Now, as illustration of the above representations, we prove that for every two natural numbers x and y

$$|\mu(x).\mu(y)| \geq |\mu(x.y)|. \tag{1}$$

Really, if x or y is not square-free, then

$$|\mu(x).\mu(y)| = \left|(-1)^{\omega(x)+\omega(y)}. \left[\frac{\underline{mult}(x)}{x}\right] . \left[\frac{\underline{mult}(y)}{y}\right]\right| = \left|\left[\frac{\underline{mult}(x)}{x}\right] . \left[\frac{\underline{mult}(y)}{y}\right]\right| = 0$$

$$= \left|\left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = \left|(-1)^{\omega(x.y)}. \left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = |\mu(x.y)|,$$

because x or y has at least one prime divisor p that has an order higher than 1, while p appears in $\underline{mult}(x)$ or $\underline{mult}(y)$, and in $\underline{mult}(x.y)$ only once. Therefore, in this case (1) is valid.

If x and y are both square-free, then

$$|\mu(x).\mu(y)| = \left|(-1)^{\omega(x)+\omega(y)}. \left[\frac{\underline{mult}(x)}{x}\right] . \left[\frac{\underline{mult}(y)}{y}\right]\right| = \left|\left[\frac{\underline{mult}(x)}{x}\right] . \left[\frac{\underline{mult}(y)}{y}\right]\right| = 1$$

$$\begin{cases} = \left|\left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = \left|(-1)^{\omega(x.y)}. \left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = |\mu(x.y)|, & \text{if } \underline{set}(x) \cap \underline{set}(y) = \emptyset \\ > 0 = \left|\left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = \left|(-1)^{\omega(x.y)}. \left[\frac{\underline{mult}(x.y)}{x.y}\right]\right| = |\mu(x.y)|, & \text{if } \underline{set}(x) \cap \underline{set}(y) \neq \emptyset \end{cases}.$$

Therefore, (1) is also valid.

Similar illustrations can be given with the other representations of μ.

References

[1] Mitrinovic, D., J. Sándor, *Handbook of Number Theory*, Kluwer Academic Publishers, 1996.

[2] Nagell, T., *Introduction to Number Theory*, John Wiley & Sons, New York, 1950.