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Note on Legendre symbols connecting with certain infinite series

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Abstract: Some applications of Legendre symbols connecting with certain infinite series are studied. In addition, some identities for Dirichlet series through Legendre symbols and Hurwitz zeta function are given.
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1 Introduction

In number theory, the Legendre symbol is a completely multiplicative function with values 1, -1, 0. Let p be any odd prime, if $n \not\equiv 0(\text{mod} p)$ Legendre symbol $(n|p)$ is defined as follows[1, p.179]

$$(n|p) = \begin{cases} +1 & \text{If } nRp \\ -1 & \text{If } n\overline{R}p \end{cases}. \tag{1.1}$$

where nRp denote n is a quadratic residue mod p and $n\overline{R}p$ denote n is a quadratic nonresidue mod p. If $n \equiv 0(\text{mod} p)$, we define $(n|p) = 0$. Recently, some properties and applications of Mobius function and a new arithmetic function connecting with infinite series, infinite products and partition of an integer are studied in Ref[2]. The purpose of this work is to study some applications of Legendre symbols connecting with infinite series of rational functions and Dirichlet series.

2 Legendre symbols connecting with infinite series

Lemma 2.1. *For any odd prime* p *and* $|x| < 1$,

$$F_p(x) = \sum_{k=1}^{\infty}(k|p)x^k = \frac{1}{1-x^p}\sum_{m=1}^{p-1}(m|p)x^m. \tag{2.1}$$

Proof. The Legendre symbol $(k|p)$ is periodic. Then

$$\sum_{k=1}^{\infty}(k|p)x^k = \sum_{k=0}^{\infty}\sum_{m=1}^{p-1}(m|p)x^{kp+m}.$$

Rearranging right hand side of above equation, gives

$$\sum_{k=1}^{\infty}(k|p)x^k = \sum_{m=1}^{p-1}(m|p)x^m\sum_{k=0}^{\infty}x^{kp}.$$

After simplification, completes the Lemma. □

Example 2.2. Let $p = 3, 5$ and 7 in (2.1). Then

$$F_3(x) = \frac{x(1-x)}{1-x^3}.$$

$$F_5(x) = \frac{x(1-x)(1-x^2)}{1-x^5}.$$

$$F_7(x) = \frac{x(1-x)(1+2x+x^2+2x^3+x^4)}{1-x^7}.$$

Theorem 2.3. *For an odd prime* p *and* $|x| < 1$,

$$x = \sum_{k=1}^{\infty}\mu(k)(k|p)F_p(x^k). \tag{2.2}$$

Proof. Using Lemma 2.1, that

$$\sum_{k=1}^{\infty}\mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty}\mu(k)(k|p)\sum_{n=1}^{\infty}(n|p)x^{kn}. \tag{2.3}$$

Rearranging above equation, gives

$$\sum_{k=1}^{\infty}\mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty}x^k\sum_{d|k}\mu(d)(d|p)(k/d|p).$$

Since $(d|p)(k/d|p) = (k|p)$,

$$\sum_{k=1}^{\infty}\mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty}x^k(k|p)\sum_{d|k}\mu(d).$$

Since $\sum_{d|1}\mu(d) = 1$, $\sum_{d|k}\mu(d) = 0 (k \geq 2)$ and $(1|p) = 1$, this completes the theorem. □

Theorem 2.4. *Let p be any odd prime and c be an arithmetic function. If $P(x) = \sum_{k=1}^{\infty} c(k)x^k$, then*

$$P(x) = \sum_{k=1}^{\infty} a(k)F_p(x^k). \tag{2.4}$$

where

$$a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d). \tag{2.5}$$

Proof. Consider

$$P(x) = \sum_{k=1}^{\infty} c(k)x^k. \tag{2.6}$$

Using Theorem 2.3, that

$$P(x) = \sum_{k=1}^{\infty} c(k) \sum_{m=1}^{\infty} \mu(m)(m|p)F_p(x^{mk}).$$

Rearranging above equation, gives

$$P(x) = \sum_{k=1}^{\infty} F_p\left(x^k\right) \sum_{d|k} \mu(d)(d|p)c(k/d).$$

Setting $a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d)$ in the above equation, completes the theorem. $\square$

Corollary 2.5. *Let $P(x) = \sum_{k=1}^{\infty} f(k)(k|p)x^k$ and p be an odd prime. Then*

$$P(x) = \sum_{k=1}^{\infty} (k|p)b(k)F_p(x^k). \tag{2.7}$$

where

$$b(k) = \sum_{d|k} \mu(d)f(k/d). \tag{2.8}$$

Proof. Let $c(k) = (k|p)f(k)$ in Theorem 2.4. Then (2.5) becomes

$$a(k) = \sum_{d|k} \mu(d)(d|p)(k/d|p)f(k/d).$$

Since $(d|p)(k/d|p) = (k|p)$,

$$a(k) = (k|p) \sum_{d|k} \mu(d)f(k/d).$$

Setting $b(k) = \sum_{d|k} \mu(d)f(k/d)$, then

$$a(k) = (k|p)b(k). \tag{2.9}$$

Substituting (2.9) in (2.4), completes the proof. $\square$

Example 2.6. Let $c(k) = 1/k^s$ in Theorem 2.4. Then

$$\Phi(x, s, 1) = \sum_{k=1}^{\infty} a(k) F_p(x^k). \tag{2.10}$$

where

$$a(k) = \frac{1}{k^s} \sum_{d|k} \mu(d)(d|p)d^s$$

and $\Phi(x, s, b)$ is Lerch zeta function defined by

$$\Phi(x, s, b) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+b)^s}.$$

Example 2.7. The following identities are some of examples of Corollary 2.5.

1. Let $f(k) = k$ in Corollary 2.5, then

$$P(x) = \sum_{k=1}^{\infty} k(k|p)x^k = xD\left[F_p(x)\right].$$

Also $b(k) = \sum_{d|k} \mu(d)\frac{k}{d} = \varphi(k)$, where φ is Euler totient function. Hence

$$xD\left[F_p(x)\right] = \sum_{k=1}^{\infty} (k|p)\varphi(k)F_p(x^k).$$

where D differential operator denotes differentiation with respect to x.

For instance, if $p = 3$

$$\frac{1-x^2}{1+x+x^2} = \varphi(1)\frac{1-x}{1-x^3} + \varphi(2)\frac{x(1-x^2)}{1-x^6} + \varphi(5)\frac{x^4(1-x^5)}{1-x^{15}} - \dots.$$

2. Let $f(k) = 1/k$, then

$$P(x) = \sum_{k=1}^{\infty} (k|p)\frac{x^k}{k} = \int_0^{\infty} F_p(xe^{-t})dt.$$

Also

$$b(k) = \frac{1}{k}\sum_{d|k} \mu(d)d = \frac{\varphi^{-1}(k)}{k},$$

where φ^{-1} is inverse of Euler totient function with respect to convolution. Hence

$$\int_0^{\infty} F_p(xe^{-t})dt = \sum_{k=1}^{\infty} (k|p)\frac{\varphi^{-1}(k)}{k}F_p(x^k).$$

For instance,

$$\int_0^{\infty} \frac{xe^{-t}(1-xe^{-t})}{1-x^3e^{-3t}}dt = \varphi^{-1}(1)\frac{x(1-x)}{1-x^3} + \frac{\varphi^{-1}(2)}{2}\frac{x^2(1-x^2)}{1-x^6} + \dots.$$

3 Some identities for Dirichlet series through Legendre symbols

Theorem 3.1. *Let $Re(s) > 1$, p be any odd prime and c be an arithmetic function. Then,*

$$\sum_{k=1}^{\infty} \frac{c(k)}{k^s} = \frac{1}{p^s} \sum_{m=1}^{p-1} (m|p)\zeta(s, m/p) \sum_{k=1}^{\infty} \frac{a(k)}{k^s}. \tag{3.1}$$

where $a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d)$.

Proof. For $|x| < 1$, let

$$P(x) = \sum_{k=1}^{\infty} c(k)x^k. \tag{3.2}$$

Replace x by e^{-t}, multiply by t^{s-1} for $Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} c(k) \int_0^{\infty} t^{s-1} e^{-kt} dt.$$

Since $\int_0^{\infty} t^{s-1} e^{-kt} dt = \frac{\Gamma(s)}{k^s}$,

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \Gamma(s) \sum_{k=1}^{\infty} \frac{c(k)}{k^s}. \tag{3.3}$$

Similarly, replace x by e^{-t} in (2.4), multiply by t^{s-1} for $Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} a(k) \int_0^{\infty} t^{s-1} F_p(e^{-kt}) dt$$

Using (2.1). gives

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} a(k) \int_0^{\infty} \frac{t^{s-1}}{1 - e^{-pkt}} \sum_{m=1}^{p-1} (m|p) e^{-mkt} dt$$

Rearranging right hand side of above equation, yields

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} a(k) \sum_{m=1}^{p-1} (m|p) \int_0^{\infty} \frac{t^{s-1}}{1 - e^{-pkt}} e^{-mkt} dt.$$

Using solution of the integral [2, p. 349]

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \frac{\Gamma(s)}{p^s} \sum_{k=1}^{\infty} \frac{a(k)}{k^s} \sum_{m=1}^{p-1} (m|p)\zeta(s, m/p). \tag{3.4}$$

Comparing (3.3) and (3.4), completes the theorem. □

Corollary 3.2. *For $Re(s) > 1$ and odd prime p,*

$$\sum_{k=1}^{\infty}(k|p)\frac{f(k)}{k^s} = \frac{1}{p^s}\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)\sum_{k=1}^{\infty}(k|p)\frac{b(k)}{k^s}. \tag{3.5}$$

where $b(k) = \sum_{d|k}\mu(d)f(k/d)$.

Proof. Let $c(k) = (k|p)f(k)$, then $a(k) = (k|p)b(k)$. This completes the corollary. □

Example 3.3. Let $f(k) = 1$ in corollary 3.2. Then

$$\sum_{k=1}^{\infty}\frac{(k|p)}{k^s} = \frac{1}{p^s}\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p). \tag{3.6}$$

The following Dirichlet series are obtained by taking $f(k) = k, k^n, 1/k$ in (3.5) and using the identity (3.6), that

$$\sum_{k=1}^{\infty}(k|p)\frac{\varphi(k)}{k^s} = p\frac{\sum_{m=1}^{p-1}(m|p)\zeta(s-1,m/p)}{\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)}.$$

$$\sum_{k=1}^{\infty}(k|p)\frac{J_n(k)}{k^s} = p^n\frac{\sum_{m=1}^{p-1}(m|p)\zeta(s-n,m/p)}{\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)}.$$

$$\sum_{k=1}^{\infty}(k|p)\frac{\varphi^{-1}(k)}{k^{s+1}} = \frac{1}{p}\frac{\sum_{m=1}^{p-1}(m|p)\zeta(s+1,m/p)}{\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)}.$$

where J_n is Jordan totient function. Let $f(k) = \mu(k)/k^\alpha$ in (3.5). Then using the identity $\sigma_\alpha^{-1}(k) = \sum_{d|k}d^\alpha\mu(d)\mu(k/d)$

$$\sum_{k=1}^{\infty}\mu(k)\frac{(k|p)}{k^{s+\alpha}} = \frac{1}{p^s}\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)\sum_{k=1}^{\infty}(k|p)\frac{\sigma_\alpha^{-1}(k)}{k^{s+\alpha}}.$$

where σ_α is the divisor function. Using (3.6), gives the following identity

$$\sum_{k=1}^{\infty}(k|p)\mu(k)\frac{\sigma_\alpha(k)}{k^{s+\alpha}} = p^{-(2s+\alpha)}\sum_{k=1}^{\infty}(k|p)\frac{\Lambda(k)}{k^s} = \log p - \frac{\sum_{m=1}^{p-1}(m|p)\zeta'(s,m/p)}{\sum_{m=1}^{p-1}(m|p)\zeta(s,m/p)}. \tag{3.7}$$

where Λ is van Mangoldt function and $d\zeta(s,m/p)/ds = \zeta'(s,m/p)$.

Theorem 3.4. *For any odd prime p,*

$$(k|p) = \frac{1}{k!}F_p^{(k)}(0). \tag{3.8}$$

$$(k|p) = \frac{\pi}{r^k}\int_0^{2\pi}C_p(r,\theta)\cos k\theta d\theta = \frac{\pi}{r^k}\int_0^{2\pi}S_p(r,\theta)\sin k\theta d\theta. \tag{3.9}$$

Where $C_p(r,\theta) = \sum_{k=1}^{\infty}(k|p)r^k\cos k\theta$ and $S_p(r,\theta) = \sum_{k=1}^{\infty}(k|p)r^k\sin k\theta$

Proof. Differentiating (2.1) with respect to k and setting $x = 0$, gives (3.9). For $|r| < 1$, $0 \leq \theta \leq 2\pi$ and odd prime p, consider

$$F_p(re^{i\theta}) = C_p(r,\theta) + iS_p(r,\theta). \tag{3.10}$$

Using (2.1), comparing real and imaginary parts, then

$$C_p(r,\theta) = \sum_{k=1}^{\infty}(k|p)r^k \cos k\theta. \tag{3.11}$$

$$S_p(r,\theta) = \sum_{k=1}^{\infty}(k|p)r^k \sin k\theta. \tag{3.12}$$

Multiply (3.12) and (3.13) by $\cos k\theta$ and $\sin k\theta$, integrating on $(0, 2\pi)$, gives (3.10). □

Remark 3.5. Squaring and adding (3.12) and (3.13) and then integrating on $[0, 2\pi]$, then

$$\int_0^{2\pi} \left(C_p(r,\theta)^2 + S_p(r,\theta)^2\right) d\theta = \frac{1}{\pi}\sum_{k=1}^{\infty}(k|p)^2 r^{2k}.$$

Since $(k|p)^2 = 1$,

$$\sum_{k=1}^{\infty}(k|p)^2 r^{2k} = \frac{r^2}{1-r^2} - \frac{r^{2p}}{1-r^{2p}}.$$

Then using (3.11), we obtain for odd prime p and $|r| < 1$,

$$\int_0^{2\pi} |F_p\left(re^{i\theta}\right)|^2 d\theta = \frac{1}{\pi}\left(\frac{r^2}{1-r^2} - \frac{r^{2p}}{1-r^{2p}}\right). \tag{3.13}$$

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