Note on Legendre symbols
connecting with certain infinite series

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Abstract: Some applications of Legendre symbols connecting with certain infinite series are studied. In addition, some identities for Dirichlet series through Legendre symbols and Hurwitz zeta function are given.

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1 Introduction

In number theory, the Legendre symbol is a completely multiplicative function with values 1, -1, 0. Let $p$ be any odd prime, if $n \not\equiv 0 \pmod{p}$ Legendre symbol $(n|p)$ is defined as follows[1, p.179]

$$\left(\frac{n}{p}\right) = \begin{cases} +1 & \text{If } nR_p \\ -1 & \text{If } nR_p \end{cases}.$$  (1.1)

where $nR_p$ denote $n$ is a quadratic residue mod $p$ and $nR_p$ denote $n$ is a quadratic nonresidue mod $p$. If $n \equiv 0 \pmod{p}$, we define $(n|p) = 0$. Recently, some properties and applications of Mobius function and a new arithmetic function connecting with infinite series, infinite products and partition of an integer are studied in Ref[2]. The purpose of this work is to study some applications of Legendre symbols connecting with infinite series of rational functions and Dirichlet series.
Legendre symbols connecting with infinite series

Lemma 2.1. For any odd prime \( p \) and \( |x| < 1 \),

\[
F_p(x) = \sum_{k=1}^{\infty} (k|p)x^k = \frac{1}{1-x^p} \sum_{m=1}^{p-1} (m|p)x^m.
\] (2.1)

Proof. The Legendre symbol \((k|p)\) is periodic. Then

\[
\sum_{k=1}^{\infty} (k|p)x^k = \sum_{k=0}^{\infty} \sum_{m=1}^{p-1} (m|p)x^{kp+m}.
\]

Rearranging right hand side of above equation, gives

\[
\sum_{k=1}^{\infty} (k|p)x^k = \sum_{m=1}^{p-1} (m|p)x^m \sum_{k=0}^{\infty} x^{kp}.
\]

After simplification, completes the Lemma.

Example 2.2. Let \( p = 3, 5 \) and \( 7 \) in (2.1). Then

\[
F_3(x) = \frac{x(1-x)}{1-x^3}.
\]

\[
F_5(x) = \frac{x(1-x)(1-x^2)}{1-x^5}.
\]

\[
F_7(x) = \frac{x(1-x)(1+2x+x^2+2x^3+x^4)}{1-x^7}.
\]

Theorem 2.3. For an odd prime \( p \) and \( |x| < 1 \),

\[
x = \sum_{k=1}^{\infty} \mu(k)(k|p)F_p(x^k).
\] (2.2)

Proof. Using Lemma 2.1, that

\[
\sum_{k=1}^{\infty} \mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty} \mu(k)(k|p) \sum_{n=1}^{\infty} (n|p)x^{kn}.
\] (2.3)

Rearranging above equation, gives

\[
\sum_{k=1}^{\infty} \mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty} x^k \sum_{d|k} \mu(d)(d|p)(k/d|p).
\]

Since \((d|p)(k/d|p) = (k|p)\),

\[
\sum_{k=1}^{\infty} \mu(k)(k|p)F_p(x^k) = \sum_{k=1}^{\infty} x^k (k|p) \sum_{d|k} \mu(d).
\]

Since \( \sum_{d|1} \mu(d) = 1, \sum_{d|k} \mu(d) = 0(k \geq 2) \) and \((1|p) = 1\), this completes the theorem. \( \square \)
Theorem 2.4. Let $p$ be any odd prime and $c$ be an arithmetic function. If $P(x) = \sum_{k=1}^{\infty} c(k)x^k$, then

$$P(x) = \sum_{k=1}^{\infty} a(k)F_p(x^k).$$

(2.4)

where

$$a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d).$$

(2.5)

Proof. Consider

$$P(x) = \sum_{k=1}^{\infty} c(k)x^k.$$  

(2.6)

Using Theorem 2.3, that

$$P(x) = \sum_{k=1}^{\infty} c(k)\sum_{m=1}^{\infty} \mu(m)(m|p)F_p(x^{mk}).$$

Rearranging above equation, gives

$$P(x) = \sum_{k=1}^{\infty} F_p(x^k)\sum_{d|k} \mu(d)(d|p)c(k/d).$$

Setting $a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d)$ in the above equation, completes the theorem. 

Corollary 2.5. Let $P(x) = \sum_{k=1}^{\infty} f(k)(k|p)x^k$ and $p$ be an odd prime. Then

$$P(x) = \sum_{k=1}^{\infty} (k|p)b(k)F_p(x^k).$$

(2.7)

where

$$b(k) = \sum_{d|k} \mu(d)f(k/d).$$

(2.8)

Proof. Let $c(k) = (k|p)f(k)$ in Theorem 2.4. Then (2.5) becomes

$$a(k) = \sum_{d|k} \mu(d)(d|p)(k/d|p)f(k/d).$$

Since $(d|p)(k/d|p) = (k|p)$,

$$a(k) = (k|p)\sum_{d|k} \mu(d)f(k/d).$$

Setting $b(k) = \sum_{d|k} \mu(d)f(k/d)$, then

$$a(k) = (k|p)b(k).$$

(2.9)

Substituting (2.9) in (2.4), completes the proof. 

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Example 2.6. Let $c(k) = 1/k^s$ in Theorem 2.4. Then

$$\Phi(x, s, 1) = \sum_{k=1}^{\infty} a(k) F_p(x^k).$$

(2.10)

where

$$a(k) = \frac{1}{k^s} \sum_{d|k} \mu(d)(d|p)d^s$$

and $\Phi(x, s, b)$ is Lerch zeta function defined by

$$\Phi(x, s, b) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+b)^s}.$$ 

Example 2.7. The following identities are some of examples of Corollary 2.5.

1. Let $f(k) = k$ in Corollary 2.5, then

$$P(x) = \sum_{k=1}^{\infty} k(k|p)x^k = x D [F_p(x)] .$$

Also $b(k) = \sum_{d|k} \mu(d)\frac{k}{d} = \varphi(k)$, where $\varphi$ is Euler totient function. Hence

$$x D [F_p(x)] = \sum_{k=1}^{\infty} (k|p)\varphi(k)F_p(x^k).$$

where $D$ differential operator denotes differentiation with respect to $x$.

For instance, if $p = 3$

$$\frac{1 - x^2}{1 + x + x^2} = \varphi(1) \frac{1 - x}{1 - x^3} + \varphi(2) \frac{x(1 - x^2)}{1 - x^6} + \varphi(5) \frac{x^4(1 - x^5)}{1 - x^{15}} - \ldots .$$

2. Let $f(k) = 1/k$, then

$$P(x) = \sum_{k=1}^{\infty} (k|p)\frac{x^k}{k} = \int_{0}^{\infty} F_p(x e^{-t}) dt.$$

Also

$$b(k) = \frac{1}{k} \sum_{d|k} \mu(d)d = \frac{\varphi^{-1}(k)}{k},$$

where $\varphi^{-1}$ is inverse of Euler totient function with respect to convolution. Hence

$$\int_{0}^{\infty} F_p(x e^{-t}) dt = \sum_{k=1}^{\infty} \frac{(k|p)\varphi^{-1}(k)}{k} F_p(x^k).$$

For instance,

$$\int_{0}^{\infty} \frac{xe^{-t}(1 - xe^{-t})}{1 - x^3e^{-3t}} dt = \varphi^{-1}(1) \frac{x(1 - x)}{1 - x^3} + \frac{\varphi^{-1}(2)}{2} \frac{x^2(1 - x^2)}{1 - x^6} + \ldots .$$
3 Some identities for Dirichlet series through Legendre symbols

Theorem 3.1. Let $\Re(s) > 1$, $p$ be any odd prime and $c$ be an arithmetic function. Then,

$$\sum_{k=1}^{\infty} \frac{c(k)}{k^s} = \frac{1}{p^s} \sum_{m=1}^{p-1} (m|p) \zeta(s, m/p) \sum_{k=1}^{\infty} \frac{a(k)}{k^s}. \tag{3.1}$$

where $a(k) = \sum_{d|k} \mu(d)(d|p)c(k/d)$.

Proof. For $|x| < 1$, let

$$P(x) = \sum_{k=1}^{\infty} c(k)x^k. \tag{3.2}$$

Replace $x$ by $e^{-t}$, multiply by $t^{s-1}$ for $\Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \sum_{k=1}^{\infty} c(k) \int_0^\infty t^{s-1} e^{-kt}dt.$$ 

Since $\int_0^\infty t^{s-1} e^{-kt}dt = \frac{\Gamma(s)}{k^s}$,

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \Gamma(s) \sum_{k=1}^{\infty} \frac{c(k)}{k^s}. \tag{3.3}$$

Similarly, replace $x$ by $e^{-t}$ in (2.4), multiply by $t^{s-1}$ for $\Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \sum_{k=1}^{\infty} a(k) \int_0^\infty t^{s-1} F_p(e^{-kt})dt.$$

Using (2.1) gives

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \sum_{k=1}^{\infty} a(k) \int_0^\infty \frac{t^{s-1}}{1 - e^{-pkt}} \sum_{m=1}^{p-1} (m|p)e^{-mkt}dt.$$

Rearranging right hand side of above equation, yields

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \sum_{k=1}^{\infty} a(k) \sum_{m=1}^{p-1} (m|p) \int_0^\infty \frac{t^{s-1}}{1 - e^{-pkt}} e^{-mkt}dt.$$

Using solution of the integral [2, p. 349]

$$\int_0^\infty t^{s-1} P(e^{-t})dt = \frac{\Gamma(s)}{p^s} \sum_{k=1}^{\infty} \frac{a(k)}{k^s} \sum_{m=1}^{p-1} (m|p) \zeta(s, m/p). \tag{3.4}$$

Comparing (3.3) and (3.4), completes the theorem.
Proof. Let \( c(k) = (k|p)f(k) \), then \( a(k) = (k|p)b(k) \). This completes the corollary. \( \square \)

Example 3.3. Let \( f(k) = 1 \) in corollary 3.2. Then

\[
\sum_{k=1}^{\infty} \frac{(k|p)}{k^s} = \frac{1}{p^s} \sum_{m=1}^{p-1} (m|p)\zeta(s, m/p). \tag{3.6}
\]

The following Dirichlet series are obtained by taking \( f(k) = k, k^n, 1/k \) in (3.5) and using the identity (3.6), that

\[
\sum_{k=1}^{\infty} \frac{(k|p)\varphi(k)}{k^s} = p \sum_{m=1}^{p-1} \frac{(m|p)\zeta(s-1, m/p)}{\sum_{m=1}^{p-1} (m|p)\zeta(s, m/p)}. \tag{3.7}
\]

\[
\sum_{k=1}^{\infty} \frac{(k|p)J_n(k)}{k^s} = p^n \sum_{m=1}^{p-1} \frac{(m|p)\zeta(s-n, m/p)}{\sum_{m=1}^{p-1} (m|p)\zeta(s, m/p)}. \tag{3.8}
\]

\[
\sum_{k=1}^{\infty} \frac{(k|p)\varphi^{-1}(k)}{k^{s+1}} = \frac{1}{p} \sum_{m=1}^{p-1} \frac{(m|p)\zeta(s+1, m/p)}{\sum_{m=1}^{p-1} (m|p)\zeta(s, m/p)}. \tag{3.9}
\]

where \( J_n \) is Jordan totient function. Let \( f(k) = \mu(k)/k^\alpha \) in (3.5). Then using the identity \( \sigma_{\alpha}^{-1}(k) = \sum d^\alpha \mu(d)\mu(k/d) \)

\[
\sum_{k=1}^{\infty} \frac{(k|p)\mu(k)}{k^{s+\alpha}} = \frac{1}{p^s} \sum_{m=1}^{p-1} (m|p)\zeta(s, m/p) \sum_{k=1}^{\infty} (k|p)\frac{\sigma_{\alpha}^{-1}(k)}{k^{s+\alpha}}. \tag{3.10}
\]

where \( \sigma_{\alpha} \) is the divisor function. Using (3.6), gives the following identity

\[
\sum_{k=1}^{\infty} (k|p)\mu(k)\frac{\sigma_{\alpha}(k)}{k^{s+\alpha}} = p^{-(2s+\alpha)} \sum_{k=1}^{\infty} (k|p)\frac{\Lambda(k)}{k^s} = \log p - \frac{\sum_{m=1}^{p-1} (m|p)\zeta'(s, m/p)}{\sum_{m=1}^{p-1} (m|p)\zeta(s, m/p)}. \tag{3.11}
\]

where \( \Lambda \) is van Mangoldt function and \( d\zeta(s, m/p)/ds = \zeta'(s, m/p) \).

Theorem 3.4. For any odd prime \( p \),

\[
(k|p) = \frac{1}{k!} F_p^{(k)}(0). \tag{3.12}
\]

\[
(k|p) = \frac{\pi}{p} \int_0^{2\pi} C_p(r, \theta) \cos k\theta d\theta = \frac{\pi}{p} \int_0^{2\pi} S_p(r, \theta) \sin k\theta d\theta. \tag{3.13}
\]

Where \( C_p(r, \theta) = \sum_{k=1}^{\infty} (k|p)r^k \cos k\theta \) and \( S_p(r, \theta) = \sum_{k=1}^{\infty} (k|p)r^k \sin k\theta \)
Proof. Differentiating (2.1) with respect to \( k \) and setting \( x = 0 \), gives (3.9). For \( |r| < 1, 0 \leq \theta \leq 2\pi \) and odd prime \( p \), consider

\[ F_p(re^{i\theta}) = C_p(r, \theta) + iS_p(r, \theta). \quad (3.10) \]

Using (2.1), comparing real and imaginary parts, then

\[ C_p(r, \theta) = \sum_{k=1}^{\infty} (k|p)r^k \cos k\theta. \quad (3.11) \]
\[ S_p(r, \theta) = \sum_{k=1}^{\infty} (k|p)r^k \sin k\theta. \quad (3.12) \]

Multiply (3.12) and (3.13) by \( \cos k\theta \) and \( \sin k\theta \), integrating on \((0, 2\pi)\), gives (3.10).

Remark 3.5. Squaring and adding (3.12) and (3.13) and then integrating on \([0, 2\pi]\), then

\[ \int_0^{2\pi} (C_p(r, \theta)^2 + S_p(r, \theta)^2) \, d\theta = \frac{1}{\pi} \sum_{k=1}^{\infty} (k|p)^2 r^{2k}. \]

Since \((k|p)^2 = 1\),

\[ \sum_{k=1}^{\infty} (k|p)^2 r^{2k} = \frac{r^2}{1 - r^2} - \frac{r^{2p}}{1 - r^{2p}}. \]

Then using (3.11), we obtain for odd prime \( p \) and \( |r| < 1 \),

\[ \int_0^{2\pi} |F_p(re^{i\theta})|^2 \, d\theta = \frac{1}{\pi} \left( \frac{r^2}{1 - r^2} - \frac{r^{2p}}{1 - r^{2p}} \right). \quad (3.13) \]

References

