A Note on modified Jacobsthal and Jacobsthal–Lucas numbers

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Abstract: In this note, some relations between Jacobsthal and Jacobsthal–Lucas numbers and their respective modifications due to K. T. Atanassov [1, 2] and Y. Shang [4] are presented.

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1 Introduction

In [3], Rabago introduced the concept of circulant determinant sequence with binomial coefficients. In particular, the right-circulant determinant sequence with binomial coefficients, denoted by \( \{ R_n \} \), is defined as a sequence of the form

\[
\{ R_n \} = \left\{ \begin{array}{c|c}
1 & 1 \\
\hline
1 & 1 \\
1 & 2 \\
2 & 1 \\
\end{array} \right\}, \left\{ \begin{array}{c|c}
1 & 3 \\
\hline
3 & 1 \\
3 & 3 \\
1 & 1 \\
\end{array} \right\}, \ldots .
\]

Furthermore, in [1], the formula for the \( n \)-th term of the sequence \( R_n \), as well as the sum of the first \( n \) terms, denoted by \( RS_n \), has been shown by the author and is given by

\[
R_n = (1 + (-1)^{n-1}) 2^{n-2}
\]

and

\[
RS_n = \frac{4 \left\lfloor \frac{n+1}{2} \right\rfloor - 1}{3}.
\]

As we recall, the \( n \)-th Jacobsthal and Jacobsthal–Lucas numbers \( (n \geq 0) \) are defined by

\[
J_n = \frac{2^n - (-1)^n}{3}
\]
and
\[ j_n = 2^n + (-1)^n, \quad (2) \]
respectively. Now, if we take the sum of the first \( n \)-term of the sequence \( \{R_n\} \) with odd indices (or simply the first \((2n - 1)\) terms of the sequence) we will obtain
\[ RS_{2n-1} = \frac{4^n - 1}{3}. \]
Fortunately, the product of the first \( n \) Jacobsthal numbers and Jacobsthal-Lucas numbers is given by the same formula. Hence, we obtain the following result.

**Theorem 1.1.** Let \( J_n, j_n \) and \( RS_n \) be the \( n \)-th Jacobsthal number, the \( n \) Jacobsthal-Lucas number, and the sum of the first \( n \) terms of the circulant determinant sequence with binomial coefficients with odd indices then,
\[ J_n j_n = RS_{2n-1}. \]

On the otherhand, Atanassov [1] provide a generalization of (1) by the following formula
\[ J_n^s = \frac{s^n - (-1)^n}{s + 1}, \quad (3) \]
where \( n \geq 0 \) is a natural number and \( s \geq 0 \) is a real number. He also introduced another generalization of (1) in [2] and is given by
\[ Y_n^s = \frac{s^n - (-1)^n}{s^2 - 1}, \quad (4) \]
where \( s \neq 1 \) is a real number. He then obtained the following interesting results.

**Theorem 1.2.** For every natural number \( n \geq 0 \) and real number \( s \neq 1 \):
\[ Y_n^s = \frac{1}{s - 1} J_n^s, \quad Y_{n+2}^s = Y_n^s + s^n, \quad Y_{n+1}^s = s Y_n^s + \frac{(-1)^n}{s - 1}. \]

As an analogue to these results, Shang [4] formulated some modifications of the Jacobsthal-Lucas numbers. More precisely, he consider the following modification of (2):
\[ j_n^s = s^n + (-1)^n, \quad (5) \]
where \( n \) is a natural number and \( s \geq 0 \) is a real number. He then further extend his modification to the following form
\[ j_n^{s,t} = s^n + (-t)^n, \quad (6) \]
where \( n \) is a natural number, \( s \) and \( t \) are arbitrary real numbers. Inspired by these results, we present some relations involving Jacobsthal numbers, Jacobsthal-Lucas numbers, and their respective generalization and modification.
2 Main results

We begin by defining $J^s_{-n}$ and $j^s_{-n}$.

For every $n \in \mathbb{N}$, we let

\[
J^s_{-n} = (-1)^{n+1} J^s_n
\]

and

\[
j^s_{-n} = (-1)^{n+1} j^s_n.
\]

Throughout the following discussion we let $C^n_k = \binom{n}{k}$.

**Theorem 2.1.** For $n, l \in \mathbb{N}$ and real number $s \geq 0$, $s \neq 1$,

\[
\sum_{k=0}^{n} C^n_k j^s_{2k} = (j^s_{2l})^n + 2^n.
\]

In particular, for $l = 1$, we have

\[
\sum_{k=0}^{n} C^n_k j^s_{2k} = (j^s_{2})^n + 2^n.
\]

**Proof.** Note that

\[
(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k (s^{2l})^k + \sum_{k=0}^{n} C^n_k (-1)^k,
\]

and since

\[
\sum_{k=0}^{n} C^n_k = 2^n,
\]

then

\[
(j^s_{2l})^n = \sum_{k=0}^{n} C^n_k ((s^l)^{2k} + 1) - 2^n = \sum_{k=0}^{n} C^n_k j^s_{2k} - 2^n.
\]

We may remark that in terms of (6), we can express (9) as $j^s_{2l} - 2$.

**Theorem 2.2.** For $n, l \in \mathbb{N}$ and real number $s \geq 0$, $s \neq 1$,

\[
\sum_{k=0}^{n} C^n_k j^s_{2l} = j^s_{2l+1} - (-1)^n.
\]

More generally, we have,

\[
\sum_{k=0}^{n} C^n_k j^s_{2l} = j^s_{2l+1} - (-1)^n.
\]

**Proof.** Because

\[
(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k (s^{2l})^k + \sum_{k=0}^{n} C^n_k (-1)^k,
\]

then

\[
(s^{2l} + 1)^n = \sum_{k=0}^{n} C^n_k j^s_{2l}
\]

Hence,

\[
j^s_{2l+1} = \sum_{k=0}^{n} C^n_k j^s_{2l} + (-1)^n
\]

Thus, conclusion follows. \(\square\)

**Theorem 2.3.** For all natural number $n$ and real number $s \geq 0$, $s \neq 1$,

\[
\sum_{k=0}^{n} C^n_k j^s_{-1} = s^n.
\]
Proof. We use the fact that \(\sum_{k=0}^{n} C_k^n (-1)^k = 0\). Since \(s^n = (s - 1 + 1)^n\) then
\[
(s - 1 + 1)^n = \sum_{k=0}^{n} C_k^n (s - 1)^k + \sum_{k=0}^{n} C_k^n (-1)^k \\
= \sum_{k=0}^{n} C_k^n ((s - 1)^k + (-1)^k) \\
= \sum_{k=0}^{n} C_k^n j_k s^{s - 1}.
\]
Thus, conclusion follows. \(\Box\)

By incorporating Theorem 1.2 to the previous theorem we obtain the following corollaries.

**Corollary 2.4.** For all natural number \(n\) and real number \(s \geq 0, s \neq 1\),
\[
Y_{n+2}^s - Y_n^s = \sum_{k=0}^{n} C_k^n j_k s^{s - 1}.
\]

**Corollary 2.5.** For all natural number \(n\) and real number \(s \geq 0, s \neq 1\),
\[
J_{n+2}^s - J_n^s = (s - 1) \sum_{k=0}^{n} C_k^n j_k s^{s - 1}.
\]

In particular, for \(s = 2\), \(J_{n+2} - J_n = \sum_{k=0}^{n} C_k^n j_k^1 = 2^n\).

Using the identities in Theorem 2.2 and Theorem 2.3 we will obtain the following theorem.

**Theorem 2.6.** For all natural number \(n\) and real number \(s \geq 0, s \neq 1\),
\[
J_n^s = \frac{1}{s + 1} \left[ \sum_{k=0}^{n} C_k^n \left( j_k^{s^2} + j_k^{s^1} \right) - j_n^{s^2 + 1} \right] = (s - 1)Y_n^s.
\]

If we let \(s = 2\) we’ll obtain the following special case of the above theorem.

**Corollary 2.7.**
\[
J_n = \frac{1}{3} \left[ \sum_{k=0}^{n} C_k^n \left( j_k^{4^1} + j_k^{4^0} \right) - j_n^{4^1 + 1} \right].
\]

Theorem 2.6 can also be shown using the fact that \(\sum_{k=0}^{n} C_k^n j_k^{2l} = (s^{2l} + 1)^n\).

**Theorem 2.8.** For all natural number \(n\) and real number \(s \geq 0, s \neq 1\),
\[
J_n = \frac{1}{3} \left[ \sum_{k=0}^{n} C_k^n \left( j_k^{2l} + j_k^{2l} \right) - j_n^{s^{2l} + 1} + (j_n^{2})^n \right].
\]

**Theorem 2.9.** For all natural number \(n\) and real number \(s \geq 0, s \neq 1\),
\[
J_n = \sum_{k=0}^{n} C_k^n \left( j_k^{2l} - j_k^{2l} \right) + (j_n^{2l + 1} - (j_2^n)^n).
\]
For the proof of Theorem 2.8 and Theorem 2.9 we may use Theorem 2.1 and 2.2.

**Theorem 2.10.** For all natural number \( n \) and real number \( s \geq 0, s \neq 1 \),
\[
\sum_{m=0}^{n} \sum_{k=0}^{m} C_{m}^{n} J_{k}^{s-1} = \frac{s^{n+1} - 1}{s - 1}.
\]

*Proof.* Use Theorem 2.3.

**Theorem 2.11.** For all even natural number \( n \),
\[
J_{n} = \frac{1}{3} \left[ \left( \sum_{k=0}^{n} (-1)^{k} C_{k}^{n} s^{k} \right) - j_{n}^{s-1} \right],
\]
where \( J_{n} \) is the \( n \)-th Jacobsthal number and \( j_{n}^{s} \) is the \( n \)-th modified Jacobsthal-Lucas number.

*Proof.* The proof is straightforward. We use the binomial expansion
\[
(x + y)^n = \sum_{k=0}^{n} C_{k}^{n} x^{k} y^{n-k}.
\]

We let \( x = s \) and \( y = -1 \) obtaining
\[
(s - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} C_{k}^{n} s^{k}.
\]

Noting that \((-1)^{n-k} = (-1)^{n+k}\), we have
\[
(s - 1)^n = \sum_{k=0}^{n} (-1)^{n+k} C_{k}^{n} s^{k} + \sum_{k=0}^{n} C_{k}^{n} - 2^n.
\]

Adding \((-1)^{n}\) both sides we’ll obtain
\[
j_{n}^{s-1} = (s - 1)^n + (-1)^n = \sum_{k=0}^{n} C_{k}^{n} ((-1)^{n+k} s^{k} + 1) - (2^n - (-1)^n),
\]
and since \( n \) is even then,
\[
j_{n}^{s-1} = \sum_{k=0}^{n} (-1)^{k} C_{k}^{n} (s^{k} + (-1)^{k}) - 3j_{n}.
\]

Thus,
\[
J_{n} = \frac{1}{3} \left[ \left( \sum_{k=0}^{n} (-1)^{k} C_{k}^{n} s^{k} \right) - j_{n}^{s-1} \right],
\]
which is the desired result.

A similar result to Theorem 2.11 can be obtain for \( j_{n} \) and is stated in the following theorem.
Theorem 2.12. For all odd natural number $n$,

$$j_n = j_{n-1}^s + \sum_{k=0}^{n} (-1)^k C_k^n j_k^s,$$  \hspace{1cm} (10)

where $j_n$ is the $n$-th Jacobsthal-Lucas number and $j_n^s$ is the $n$-th modified Jacobsthal-Lucas number.

Proof. The proof is similar to the previous theorem. Again we let $x = s$ and $y = -1$ in the binomial expansion obtaining

$$(s - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} C_k^n s^n.$$  

So we have,

$$(s - 1)^n = \sum_{k=0}^{n} (-1)^{n+k} C_k^n j^s_k - \sum_{k=0}^{n} C_k^n + 2^n.$$  

Adding both sides by $(-1)^n$ and noting that $n$ is odd we have

$$j_{n-1}^s = - \sum_{k=0}^{n} (-1)^k C_k^n (s^k + (-1)^k) + j_n.$$  

It follows that,

$$j_n = j_{n-1}^s + \sum_{k=0}^{n} (-1)^k C_k^n j_k^s.$$  

This proves the theorem. \hfill $\square$

We could express (10) using (8) and is given by the following Corollary.

Corollary 2.13. $j_n = j_{n-1}^s - \sum_{k=0}^{n} C_k^n j_k^s.$

References


