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New explicit representations for the prime counting function

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Abstract: In the paper the new formulae for the prime counting function π:

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{k+1}{\sigma(k)} \right)^{k+\sqrt{k}} \right\rfloor ; \pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{k+1}{\psi(k)} \right)^{k+\sqrt{k}} \right\rfloor$$

(where σ is the sum-of-divisor function and ψ is the Dedekind's function) are proposed and proved. Also a general theorem (Theorem 1) is obtained that gives infinitely many explicit formulae for the prime counting function π (depending on arbitrary arithmetic function with strictly positive values, satisfying certain condition).

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Used denotations

$\lfloor\rfloor$ – denotes the floor function, i.e. $\lfloor x \rfloor$ denotes the largest integer that is not greater than the real non-negative number x; σ – denotes the so-called sum-of-divisor function, i.e. $\sigma(1) = 1$ and for integer $n > 1$

$$\sigma(n) = \sum_{d|n} d,$$

where $\sum\limits_{d|n}$ means that the sum is taken over all divisors d of n; ψ – denotes Dedekind's function, i.e. $\psi(1) = 1$ and for integer $n > 1$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p} \right),$$

where $\prod\limits_{p|n}$ means that the product is taken over all prime divisors p of n; π – denotes the prime counting function, i.e. for any integer $n \geq 2$, $\pi(n)$ denotes the number of primes p, satisfying the inequality $p \leq n$.

1 Introduction

In year 2001, the author (in [1]) proposed (for the first time) the following formula for the prime counting function:

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{k+1}{\sigma(k)} \right\rfloor .$$

Let θ is either σ or Dedekind's function ψ. Then it is not hard to see that for any integer $n \geq 2$ the formula

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{k+1}{\theta(k)} \right\rfloor$$

is also true.

These results have motivated us for the results obtained in the present paper.

2 Preliminary results

Lemma 1. *For any composite $k > 1$,*

$$\theta(k) \geq k + \sqrt{k}. \tag{1}$$

Proof. First we observe that for any $k \geq 1$, $\sigma(k) \geq \psi(k)$. Let $p \geq 2$ be the minimal prime divisor of k. Then $p \leq \sqrt{k}$ and from the obvious inequality

$$\theta(k) \geq k\left(1 + \frac{1}{p}\right)$$

we obtain

$$\theta(k) \geq k + \frac{k}{p} \geq k + \frac{k}{\sqrt{k}} = k + \sqrt{k}.$$

Hence (1) is true.

Lemma 1 is proved.

□

Lemma 2. *Let the sequence $\{c_k\}_{k=2}^{\infty}$ is defined by*

$$c_k \stackrel{\text{def}}{=} \left(1 - \frac{\sqrt{k}-1}{k+\sqrt{k}}\right)^{\frac{k+\sqrt{k}}{\sqrt{k}-1}}, k = 2, 3, 4 \ldots .$$

Then for any $k \geq 2$, the inequality $c_k < e^{-1}$ holds.

Proof. The validity of the assertion is checked directly for $k = 2, 3, 4, 5, 6$. For $k > 6$ the function $g(k) \stackrel{\text{def}}{=} \frac{k+\sqrt{k}}{\sqrt{k}-1}$ is strictly increasing and tends to $+\infty$. Also we have $c_k = \left(1 - \frac{1}{g(k)}\right)^{g(k)}$.

Hence, for $k > 6$ the validity of Lemma 2 holds from the fact, that the function $h(x) \stackrel{\text{def}}{=} (1 - \frac{1}{x})^x$ is strictly increasing for $x > 1$ and tends to e^{-1}.

□

3 Main results

Theorem 1. *Let f is an arithmetic function with strictly positive values. If for f there exists a composite number $T_f > 1$ such that the inequality*

$$\sum_{\substack{k=4 \\ k\text{ - composite}}}^{T_f-1} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} + \sum_{k=T_f}^{\infty} e^{-\frac{\sqrt{k}-1}{k+\sqrt{k}}f(k)} < 1 \tag{2}$$

holds, then for any integer $n \geq 2$

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} \right\rfloor. \tag{3}$$

Remark 1. For $T_f = 4$, (2) is reduced to the condition

$$\sum_{k=T_f}^{\infty} e^{-\frac{\sqrt{k}-1}{\sqrt{k}+k}f(k)} < 1.$$

Remark 2. Further we suppose that T_f is the minimal composite number satisfying (2).

Proof. For $n \leq 3$, (3) is true. Let $n \geq 4$. Since $\frac{k+1}{\theta(k)} = 1$ for prime k, it is fulfilled

$$\sum_{k=2}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} = \pi(n) + \sum_{\substack{k=4 \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)}. \tag{4}$$

Let $n < T_f$. Then $n \leq T_f - 1$. Hence:

$$\sum_{\substack{k=4 \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} \leq \sum_{\substack{k=4 \\ \text{k - composite}}}^{T_f-1} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} < \text{ (due to (2)) } < 1.$$

Therefore, (4) and the above inequality yield (3).

Let $n \geq T_f$. Then:

$$\sum_{\substack{k=4 \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} = \sum_{\substack{k=4 \\ \text{k - composite}}}^{T_f-1} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} + \sum_{\substack{k=T_f \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)}. \tag{5}$$

But

$$\sum_{\substack{k=T_f \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} < \sum_{\substack{k=T_f \\ \text{k - composite}}}^{\infty} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} < \text{ (because of Lemma 1)}$$

$$< \sum_{\substack{k=T_f \\ \text{k - composite}}}^{\infty} \left(\frac{k+1}{k+\sqrt{k}}\right)^{f(k)} < \sum_{k=T_f}^{\infty} \left(\frac{k+1}{k+\sqrt{k}}\right)^{f(k)} = \sum_{k=T_f}^{\infty} c_k^{\frac{\sqrt{k}-1}{k+\sqrt{k}}f(k)} <$$

$$\text{(because of Lemma 2) } < \sum_{k=T_f}^{\infty} e^{-\frac{\sqrt{k}-1}{k+\sqrt{k}}f(k)}. \tag{6}$$

From (5) and (6) we obtain:

$$\sum_{\substack{k=4 \\ \text{k - composite}}}^{n} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} < \sum_{\substack{k=4 \\ \text{k - composite}}}^{T_f - 1} \left(\frac{k+1}{\theta(k)}\right)^{f(k)} + \sum_{k=T_f}^{\infty} e^{-\frac{\sqrt{k}-1}{\sqrt{k}+k} f(k)} < \text{ (due to (2)) } < 1. \quad (7)$$

Now (4) and (7) yield (3).

Theorem 1 is proved.

□

The following Theorem may be considered as a Corollary from Theorem 1.

Theorem 2. *For any integer* $n \geq 2$

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{k+1}{\theta(k)}\right)^{k+\sqrt{k}} \right\rfloor . \quad (8)$$

Proof. Let $f(k) = k + \sqrt{k}$, $k = 2, 3, 4, \dots$. Below we will show that for $T_f = 18$ the condition (2) is fulfilled. This means that the inequality

$$\sum_{\substack{k=4 \\ \text{k - composite}}}^{17} \left(\frac{k+1}{\theta(k)}\right)^{k+\sqrt{k}} + \sum_{k=18}^{\infty} e^{-(\sqrt{k}-1)} < 1 \quad (9)$$

must hold.

Since it is fulfilled:

$$\sum_{k=18}^{\infty} e^{-(\sqrt{k}-1)} < e \int_{17}^{\infty} e^{-\sqrt{k}}\, dk = 2e \int_{\sqrt{17}}^{\infty} t\, e^{-t}\, dt = 2(1+\sqrt{17})e^{1-\sqrt{17}} = 0.451041\ldots < 0.46$$

and

$$\sum_{\substack{k=4 \\ \text{k - composite}}}^{17} \left(\frac{k+1}{\theta(k)}\right)^{k+\sqrt{k}} \leq \sum_{\substack{k=4 \\ \text{k - composite}}}^{17} \left(\frac{k+1}{\psi(k)}\right)^{k+\sqrt{k}} = 0.50281\ldots < 0.51,$$

then (9) holds because $0.46 + 0.51 < 1$.

Therefore, the condition (2) is verified for $f(k) = k + \sqrt{k}$ and applying Theorem 1, (8) is proved.

Theorem 2 is proved. □

References

[1] Vassilev-Missana, M. Three Formulae for n-th Prime and Six for n-th Term of Twin Primes, *Notes on Number Theory and Discrete Mathematics*, Vol. 7, 2001, No. 1, 15–20.