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On the area and volume of a certain regular solid and the Diophantine equation $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

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Abstract: In this paper, we study some elementary problems involving surface area and volume of a certain regular solid. In particular, we find integral dimensions of a rectangular prism in which its surface area and volume are numerically equal. The problem leads us in solving a specific case of the well-known Diophantine problem

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

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1 Introduction

The Diophantine equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

for $n \geq 2$ which is formally known as the *Erdős–Straus conjecture* remains unsolved in the last 65 years. Many papers has been published in relation to this problem since 1950. In his book *Unsolved Problems in Number Theory*, R. K. Guy [1] mentioned that, when the results of L. Bernstein, R. Obláth, K. Yamamoto, and L. Rosati on this problem are put together, the conjecture is verified to be true for all n, except possibly for

$$n \equiv 1, 121, 169, 289, 361, 529 \text{ mod } 840.$$

In [4], M. Monks and A. Velingker examined the solutions to (1) for the special case in which n is prime. E. J. Ionascu and A. Wilson [2], proved an extension of Mordell's theorem and formulate a conjecture which is stronger than Erdős' conjecture and in [3], H. Kishan *et. al.,* studied some cases of the Diophantine equation of the form

$$\frac{m}{n} = \sum_{j=1}^{k} \frac{1}{x_j}.$$

In this note, we study the case when $n = 8$ in which we will see later that a geometrical interpretation of the resulting equation is obtained. Furthermore, we give all solutions satisfying (1) for $n = 8$ via elementary methods.

2 Main results

In this section, we study the equation for which the area and volume of a rectangular prism are equal in terms of numerical values.

We let l, w and h be positive integers which are defined to be the dimensions of the rectangular solid being the length, width and height respectively. So,

$$lwh = 2(lw + wh + lh), \tag{2}$$

the right hand side of the equation is basically the volume of the solid and the left hand side, on the other hand, is the area. We have three possible cases:

Case 1. The solid is a cube. If $l = w = h$, then (2) would become $l^3 = 6l^2$ in which we see that the only solution is $(l, w, h) = (6, 6, 6)$.

Case 2. One pair of dimensions are equal. Suppose (WLOG) $l = w$ and $h \neq w$, then we have $l^2h = 2l^2 + 4lh$. Which implies that $l((h-2)l - 4h) = 0$. Letting $l = h + k$, we have

$$\begin{aligned} (h-2)l - 4h &= 0 \\ (h-2)(h+k) - 4h &= 0 \\ h^2 + (k-6)h - 2k &= 0. \end{aligned}$$

The last equation is quadratic in h. Hence, we have

$$h = \frac{6 - k \pm \sqrt{(k-6)^2 + 8k}}{2}.$$

Since our interest remains only on integer values, then, we have to find all values of k in which the discriminant $(k-6)^2 + 8k$ is a perfect square. That is, we find pairs (k, n) for which the equation $(k-6)^2 + 8k = n^2$ is true. So we have,

$$\begin{aligned} (k-2)^2 - n^2 &= -32 \\ (k-n-2)(k+n-2) &= -2^5 \end{aligned}$$

But since that the prime factorization of any integer is unique. Then, for $m \leq 5$, we see that

$$\begin{cases} k - n - 2 = -2^m \\ \\ k + n - 2 = 2^{5-m} \end{cases}$$

Solving for k and n, we obtain

$$k = 2^{4-m} - 2^{m-1} + 2$$

and

$$n = 2^{4-m} + 2^{m-1}.$$

Thus, for $1 \leq m \leq 4$, we have the following values for k and n.

m	k	n
1	9	9
2	4	6
3	0	6
4	–5	9

From these values of k, we have the following table for l and h.

k	$n = \sqrt{(k-6)^2 + 8k}$	$h = \frac{6-k\pm\sqrt{(k-6)^2+8k}}{2}$	$l = h + k$
9	9	3, –6	12, 3
4	6	4, –1	8, 3
0	6	6, 0	6, 0
-5	9	10, 1	5, –4

Thus, we have three solutions distinct from the solution of the first case, in particular we have $(l, w, h) = (12, 12, 3), (8, 8, 4)$, and $(5, 5, 10)$.

Now, we proceed on the last and more general case of the problem.

Case 3. No pair of dimensions are equal. To obtain all solutions of (2) in which no pair of values are equal, we use the results found on the first two cases. The idea is to make one dimension fixed and express the remaining sums into a different form. We continue this process until all other solutions are obtained. To do this we use the following results due to K. Zelator [5].

Theorem 2.1. *Let p and q be natural numbers, with p a prime. Then, the diophantine equation*

$$\frac{q}{p} = \frac{1}{x} + \frac{1}{y}$$

has a solution in $\mathbb{N}$, with $x \neq y$, if and only if $p + 1 \equiv 0 \pmod{q}$. If the above equation is solvable, then the solution is uniquely determined (up to symmetry) and given by

$$x = \frac{1}{\frac{p+1}{q}} \text{ and } y = \frac{1}{p\left(\frac{p+1}{q}\right)}.$$

Corollary 2.2. *If p and q are arbitrary positive integers with $p + 1 \equiv 0 \pmod{q}$, then* $\left\{ \frac{1}{\frac{p+1}{q}}, \frac{1}{p\left(\frac{p+1}{q}\right)} \right\}$ *is a solution to the same equation.*

Now we illustrate the idea on how to obtain the remaining solutions. We begin by expressing the sum

$$\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

as a sum of two distinct unit fractions. That is, by Theorem 2.1, we have

$$\frac{1}{6}+\frac{1}{6}=\frac{1}{4}+\frac{1}{12}.$$

It follows that, $(4, 6, 12)$ is another solution of (2). Given this solution, we apply the same method to obtain another set of solutions satisfying (2). If 4 is held fixed, then we have to express

$$\frac{1}{6}+\frac{1}{12}=\frac{1}{4}=\frac{q}{p}$$

as a sum of another two unit fractions. Since $q|(p+1)$ then

$$\frac{1}{6}+\frac{1}{12}=\frac{1}{5}+\frac{1}{20}.$$

Hence, we obtain another solution of (2) which is $(4,\ 5,\ 20)$. If 6 is held fixed, we see that we will obtain the same solution as for the first case. Similarly, if 12 is held fixed then we will have (12, 12, 3) as a result. This is the same as what we have obtained on the second case. We do the same process on $(12,\ 12,\ 3)$, $(8,\ 8,\ 4)$, and $(5,\ 5,\ 10)$ and to the resulting sets of solutions it produced until all possibilities are exhausted.

We formalize our results by the following theorem.

Theorem 2.3. *For $n = 8$, the Diophantine equation*

$$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

has the following set of solutions $\{(x, y, z)\}$,

$$\begin{aligned}\{(x,y,z)\} \quad = \quad & \{(6,6,6),(8,8,4),(5,5,10),(12,12,3),(3,7,42),\\ & (3,8,24),(3,9,18),(3,10,15),(4,5,20),(4,6,12)\}.\end{aligned}$$

In relation to the previous theorem, we have the following remark.

Remark 2.4. *The surface-area and volume of a rectangular prism are numerically equal if and only if its dimensions is one of the following:*

$$(6,6,6),(8,8,4),(5,5,10),(12,12,3),(3,7,42),$$

$$(3,8,24),(3,9,18),(3,10,15),(4,5,20),(4,6,12).$$

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