Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 4, 28–32

On the number of sums of three unit fractions

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Abstract: Rational fractions can often be expressed as the sum of three unit fractions and, generally, such a fraction can be expanded in several ways. An estimate of the maximum number of possible solutions is given. An expression for five possible solutions is given and this is used to obtain several general expansions.
Keywords: Polynomials, Unit fractions.
AMS Classification: 11D68.

1 Introduction

Many fractions (a/b, where $0 < a < b$ are integers) can be expressed as the sum of three unit fractions

$$\frac{a}{b} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \quad 0 < x \le y \le z, \tag{1}$$

where the denominators are integers [6, 7]. It is normal to require that the denominators are distinct, but I relax this constraint for reasons that will become apparent. Of course, some fractions can not be written in this form [8, 9], but where (1) does apply, there are likely to be solutions for more than one x for each of which there may be more than one (y, z) [1, 2]. This prompts one to ask how many solutions there might be and how to estimate the denominators, and, perhaps, what determines whether or not there are solutions to (1).

2 Upper limit of the number of solutions

There can be many solutions to (1) and some of them are made clearer in this form

$$((ax-b)y - bx)((ax-b)z - bx) = b^2x^2. \tag{2}$$

Five possible solutions can be obtained directly from (2), corresponding to

$$((ax-b)y - bx, (ax-b)z - bx) = (b, bx^2), (x, b^2x), (bx, bx), (x^2, b^2), (1, b^2x^2) \tag{3}$$

if $x < b$, and there are more unless both b and x are prime. The solution corresponding to (bx, bx) must yield $y = z$, which necessitates the relaxation of the usual requirement that x, y and z be distinct (1).

Since the divisors are employed in complementary pairs, the number of possible solutions is

$$N_x \le \left\lceil \frac{1}{2} d\left(b^2x^2 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}\right) \right\rceil = \left\lceil \frac{1}{2} \prod_{i=1}^{n} (\alpha_i + 1) \right\rceil, \tag{4}$$

where the p_i are the prime factors of b^2x^2 and the α_i are integers.

For example, if $x = 41$ and $b = 121$, $b^2x^2 = 11^4 \cdot 41^2$ and the divisors are the five in (3) $((11^2 = b, 11^2 \cdot 41^2 = bx^2), (41 = x, 11^4 \cdot 41 = b^2x), (11^2 \cdot 41 = bx, 11^2 \cdot 41 = bx), (41^2 = x^2, 11^4 = b^2)$ and $(1, 11^4 \cdot 41^2 = b^2x^2))$ and three others $((11, 11^3 \cdot 41^2), (11^3, 11 \cdot 41^2), (11 \cdot 41, 11^3 \cdot 41))$. The number of possible solutions can vary considerably. For example, if $b = 121$, $N_x = 8$ for $x = 41$ (listed above), but for $x = 42$ $(b^2x^2 = 2^2 \cdot 3^2 \cdot 7^2 \cdot 11^2)$ $N_x = 41$. However, some of these possibilities may not yield integer solutions. For example, if $a = 3$, $b = 121$ and $x = 41$, all of the 8 possibilities yield integer solutions, but for $x = 42$, only 15 of the 41 possibilities are solutions. Of course, N_x is an estimate of the number of possible solutions for a specific x and many different values of x may be possible, so

$$N \le \sum_{x=\lfloor b/a \rfloor+1}^{\lfloor 3b/a \rfloor} \left\lceil \frac{1}{2} d\left(b^2x^2\right) \right\rceil.$$

For example, for a = 3 and $b = 121$, there are 64 solutions for x ranging from 41 to 66 [1].

3 Determining the denominators

The five possible solutions given in (3) yield

$$\begin{aligned}(y,z) = &\left(\frac{b(x+1)}{ax-b}, \frac{bx(x+1)}{ax-b}\right), \left(\frac{x(b+1)}{ax-b}, \frac{bx(b+1)}{ax-b}\right), \left(\frac{2bx}{ax-b}, \frac{2bx}{ax-b}\right), \\ &\left(\frac{b(x+b)}{ax-b}, \frac{x(x+b)}{ax-b}\right), \left(\frac{1+bx}{ax-b}, \frac{bx(1+bx)}{ax-b}\right)\end{aligned} . \tag{5}$$

If either b or x is not prime the other solutions are easily determined in the same way. Obviously, all of the solutions depend on knowing x, but since $\lceil b/a \rceil \le x \le \lfloor 3b/a \rfloor$ it is simple, if potentially tedious, to identify appropriate values of x.

In general, whether or not b or x is prime, the solutions of (2) are

$$(y,z) = \left(\frac{bx+p}{ax-b}, \frac{bx}{p} \cdot \frac{bx+p}{ax-b}\right)$$

where p is product of some of the prime factors of bx. There is no solution if $(ax - p) \nmid (bx + p)$ or $p \nmid bx$ which might occur, for example, if $p = x^2$ (3).

The five analytical solutions in (5) can be used to obtain expansions of the form

$$\frac{a}{F_0(n)} = \frac{1}{F_1(n)} + \frac{1}{F_2(n)} + \frac{1}{F_3(n)} \tag{6}$$

where the $F_i(n)$ are polynomials in integer n with integral coefficients. As for (1), it is not possible to express every fraction in the form of (6) and it is not possible to write a general expression in this form [4]. However, there are many specific examples of these [3, 5], but as an example the explicit expansions of $3/(6n + 1)$ obtained using the five solutions in (5) are

$$\begin{aligned}\frac{3}{6n+1} &= \frac{1}{2n+1}+\frac{1}{(n+1)(6n+1)}+\frac{1}{(n+1)(2n+1)(6n+1)} \\ &= \frac{1}{2n+1}+\frac{1}{(2n+1)(3n+1)}+\frac{1}{(2n+1)(3n+1)(6n+1)} \\ &= \frac{1}{2n+1}+\frac{1}{(2n+1)(6n+1)}+\frac{1}{(2n+1)(6n+1)} \\ &= \frac{1}{2n+1}+\frac{1}{(2n+1)(4n+1)}+\frac{1}{(4n+1)(6n+1)} \\ &= \frac{1}{2n+1}+\frac{1}{6n^2+4n+1}+\frac{1}{(2n+1)(6n+1)(6n^2+4n+1)}\end{aligned} \tag{7}$$

as is easily confirmed. Of these, Schinzel [3] credited the fourth to Sierpinski and I have reported the second previously [2]. All of these expansions have the smallest x, but it is also possible to use the first solution, for example, in (5) to obtain expansions with larger x values

$$\begin{aligned}\frac{3}{6n+1} &= \frac{1}{3n+1}+\frac{1}{6n+1}+\frac{1}{(3n+1)(6n+1)} \\ &= \frac{1}{4n+1}+\frac{1}{4n+1}+\frac{1}{(4n+1)(6n+1)} \\ &= \frac{1}{6n+1}+\frac{1}{6n+1}+\frac{1}{6n+1}\end{aligned}$$

of which I have previously reported the first [2] and the last is trivial.

Similarly, (5) also provides an easy means of generating expansions related to (7):

$$\frac{3}{6n+5} = \frac{1}{2n+2}+\frac{1}{(2n+3)(6n+5)}+\frac{1}{(2n+2)(2n+3)(6n+5)}$$

$$\frac{3}{6n+4} = \frac{1}{2n+2}+\frac{1}{(2n+3)(3n+2)}+\frac{1}{(2n+2)(2n+3)(3n+2)}$$

$$\frac{3}{6n+2} = \frac{1}{2n+1}+\frac{1}{4(n+1)(3n+1)}+\frac{1}{4(n+1)(2n+1)(3n+1)}$$

$$\frac{3}{6n-1} = \frac{1}{3n}+\frac{1}{6n-1}+\frac{1}{3n(6n-1)}$$

$$\frac{3}{6n-2} = \frac{1}{3n-1}+\frac{1}{6n}+\frac{1}{6n(3n-1)}$$

$$\frac{3}{6n-4} = \frac{1}{2n-1}+\frac{1}{4n(3n-2)}+\frac{1}{4n(2n-1)(3n-2)}$$

$$\frac{3}{6n-5}=\frac{1}{3n-2}+\frac{1}{6n-5}+\frac{1}{(3n-2)(6n-5)}.$$

4 More general expansions

Equation (5) can be used to generate much more general expressions. For example, $((ax-b)y-bx,(ax-b)z-bx)=(b,bx^2)$ yields

$$\frac{3}{2An+5}=\frac{1}{An+3}+\frac{1}{2An+5}+\frac{1}{(An+3)(2An+5)}$$

$$\frac{3}{2An+3}=\frac{1}{An+2}+\frac{1}{2An+3}+\frac{1}{(An+2)(2An+3)}$$

$$\frac{3}{2An+1}=\frac{1}{An+1}+\frac{1}{2An+1}+\frac{1}{(An+1)(2An+1)}$$

$$\frac{3}{2An-1}=\frac{1}{An}+\frac{1}{2An-1}+\frac{1}{An(2An-1)}$$

$$\frac{3}{2An-3}=\frac{1}{An-1}+\frac{1}{2An-3}+\frac{1}{(An-1)(2An-3)}$$

$$\frac{3}{2An-5}=\frac{1}{An-2}+\frac{1}{2An-5}+\frac{1}{(An-2)(2An-5)}$$

for integer $A \geq 1$ and $n \geq 1$, which can be summarised as

$$\frac{3}{2An+2m+1}=\frac{1}{An+m+1}+\frac{1}{2An+2m+1}+\frac{1}{(An+m+1)(2An+2m+1)} \tag{8}$$

$\{m \in \mathbb{Z}: An+m > 1\}$.

Using $((ax-b)y-bx,(ax-b)z-bx)=(x,b^2x)$ a related general solution is

$$\frac{3}{2An+2m}=\frac{1}{An+m}+\frac{1}{2An+2m+1}+\frac{1}{2(An+m)(2An+2m+1)}. \tag{9}$$

Finally, expressions in which a is not fixed can be obtained from (5)

$$\frac{a}{2an+a-1}=\frac{1}{2n+1}+\frac{1}{2(n+1)(2an+a-1)}+\frac{1}{2(n+1)(2n+1)(2an+a-1)}$$

$$\frac{a}{2an+a-2}=\frac{1}{2n+1}+\frac{1}{(n+1)(2an+a-2)}+\frac{1}{(n+1)(2n+1)(2an+a-2)}$$

$$\frac{a}{2an+3a-1}=\frac{1}{2n+3}+\frac{1}{(2n+4)(2an+3a-1)}+\frac{1}{(2n+3)(2n+4)(2an+3a-1)}$$

and the more general expression for integer B

$$\frac{a}{2an+Ba-1}=\frac{1}{2n+B}+\frac{1}{(2n+B+1)(2an+Ba-1)}+\frac{1}{(2n+B)(2n+B+1)(2an+Ba-1)}. \tag{10}$$

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