Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 4, 77–85

Exact formulae for the prime counting function

Mladen Vassilev–Missana

5 V. Hugo Str, 1124 Sofia, Bulgaria
e-mail: missana@abv.bg

Abstract: In the paper the following explicit formulae for the prime counting function π are proposed and proved:

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)} \right)^{\theta(k)} \right\rfloor ; \pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)} \right)^{k-1} \right\rfloor ,$$

where τ is the number-of-divisors function, θ is either the sum-of-divisors function σ or Dedekind function ψ and $\lfloor\rfloor$ is the floor function. Also an important general theorem (see Theorem 5) which gives an exact formula (depending on an arbitrary arithmetic function with strictly positive values, satisfying certain condition) for the prime counting function π is formulated and proved. This theorem generalizes all other main results in the paper.

Keywords: Prime number, Composite number, Arithmetic function.
AMS Classification: 11A25, 11A41.

Used denotations

$\lfloor\rfloor$ – denotes the floor function, i.e. $\lfloor x \rfloor$ denotes the largest integer that is not greater than the real non-negative number x; σ – denotes the so-called sum-of-divisor function, i.e. $\sigma(1) = 1$ and for integer $n > 1$

$$\sigma(n) = \sum_{d|n} d,$$

where $\sum\limits_{d|n}$ means that the sum is taken over all divisors d of n; τ – denotes the number-of-divisors function, i.e. $\tau(1) = 1$ and for integer $n > 1$

$$\tau(n) = \sum_{d|n} 1;$$

ψ – denotes Dedekind's function, i.e. $\psi(1) = 1$ and for integer $n > 1$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where $\prod_{p|n}$ means that the product is taken over all prime divisors p of n; π – denotes the prime counting function, i.e. for any integer $n \geq 2$, $\pi(n)$ denotes the number of primes p, satisfying the inequality $p \leq n$.

1 Preliminary results

First, we need some results.

Lemma 1. *Let $k > 1$ be a composite number. Then the inequality*

$$\psi(k) \geq k + \sqrt{k} \tag{1}$$

holds. Equality is only possible when $k = p^2$, with p – prime number.

Proof. Let $p \geq 2$ be the minimal prime divisor of k. Then $p \leq \sqrt{k}$ and from the obvious inequality

$$\psi(k) \geq k \left(1 + \frac{1}{p}\right)$$

we obtain

$$\psi(k) \geq k + \frac{k}{p} \geq k + \frac{k}{\sqrt{k}} = k + \sqrt{k}.$$

Hence (1) is proved. Equality is possible only for $p = \sqrt{k}$, i.e. for $k = p^2$.

Lemma 1 is proved.

□

Since for composite $k \geq 1$ we have $\sigma(k) \geq \psi(k)$ and for $k = p^2$ (p – prime) we have $\sigma(k) > \psi(k)$, the following result is obtained from Lemma 1:

Lemma 2 ([1, p.180, Exercise 4])**.** *Let $k > 1$ be a composite number. Then the inequality*

$$\sigma(k) > k + \sqrt{k}$$

holds.

Lemma 3 ([2, p. 51, Problem 3.58])**.** *Let $k > 6$ be a composite number. Then the inequality*

$$\frac{k\,\tau(k)}{\sigma(k)} > 2$$

holds.

From the above inequality we obtain

Lemma 4. *Let $k > 6$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < \frac{k}{\sigma(k)}$$

holds.

Further we need

Lemma 5. *Let $k > 1$ be integer. Then the inequality*

$$\frac{k}{\sigma(k)} < e^{-\frac{\sigma(k)-k}{\sigma(k)}}$$

holds.

Proof. Let $f(x) \stackrel{\text{def}}{=} \left(1 - \frac{1}{x}\right)^x$. Then f is strictly monotonously increasing function in the interval $(1, +\infty)$ and $\lim\limits_{x\to\infty} f(x) = e^{-1}$.

Let $b_k \stackrel{\text{def}}{=} \left(1 - \frac{1}{\frac{\sigma(k)}{\sigma(k)-k}}\right)^{\frac{\sigma(k)}{\sigma(k)-k}}$, $k = 2, 3, 4, \ldots$. From the mentioned above: $b_k < e^{-1}$. Hence, for any integer $k > 1$ we have:

$$\frac{k}{\sigma(k)} = \frac{\sigma(k) - (\sigma(k) - k)}{\sigma(k)} = 1 - \frac{\sigma(k) - k}{\sigma(k)} = b_k^{\frac{\sigma(k)-k}{\sigma(k)}} < e^{-\frac{\sigma(k)-k}{\sigma(k)}}$$

and the lemma is proved. □

As a corollary from Lemma 2, Lemma 4 and Lemma 5 we obtain

Lemma 6. *Let $k > 6$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < e^{-\frac{\sqrt{k}}{\sigma(k)}} \tag{2}$$

holds.

Lemma 4 and Lemma 5 yield

Lemma 7. *Let $k \geq 8$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < e^{-\frac{\sigma(k)-k}{\sigma(k)}} \tag{3}$$

holds.

From this we obtain

Lemma 8. *Let $k \geq 8$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < e^{-\frac{\psi(k)-k}{\psi(k)}} \tag{4}$$

holds.

Proof. The inequality $\psi(k) \leq \sigma(k)$ implies:

$$1 - \frac{k}{\psi(k)} \leq 1 - \frac{k}{\sigma(k)}.$$

Hence

$$\frac{\psi(k)-k}{\psi(k)} \leq \frac{\sigma(k)-k}{\sigma(k)}.$$

From (3) and from the last inequality, (4) follows and the lemma is proved.

□

From Lemma 1 (see (1)) and from Lemma 8 (see (4)) we obtain

Lemma 9. *Let $k \geq 8$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < e^{-\frac{\sqrt{k}}{\psi(k)}} \tag{5}$$

holds.

Lemma 9 suggests that the following assertion is true.

Lemma 10. *Let $k \geq 4$ be a composite number. Then the inequality*

$$\frac{2}{\tau(k)} < e^{-\frac{1}{\sqrt{k}+1}} \tag{6}$$

holds.

Proof. Let $k \geq 4$ be a composite number. Then

$$\frac{2}{\tau(k)} \leq \frac{2}{3} = 0.(6) < 0.716531\ldots = e^{-\frac{1}{\sqrt{4}+1}} < e^{\frac{-1}{\sqrt{k}+1}}$$

and the proof of Lemma 10 is finished.

□

2 Main results

Now we are ready to formulate and prove the first main result of the paper.

Theorem 1. *For any integer $n \geq 2$ the formula*

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{\sigma(k)} \right\rfloor \tag{7}$$

is valid.

Proof. Let $n = 2, 3, 4, 5, 6, 7$. Then the direct check shows that (7) is true. Let $n \geq 8$ be an integer. Since for prime k we have $\frac{2}{\tau(k)} = 1$, we obtain

$$\sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{\sigma(k)} = \pi(n) + \left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + \sum_{\substack{8 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{\sigma(k)}. \tag{8}$$

From (8) it is seen that to prove the theorem it remains only to establish that:

$$\left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + \sum_{\substack{8 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{\sigma(k)} < 1. \tag{9}$$

From (2) it follows that

$$\left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + \sum_{\substack{8 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{\sigma(k)} < \left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + \sum_{k=8}^{\infty} e^{-\sqrt{k}}.$$

Therefore, to prove (9) it is enough to prove the inequality

$$\left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + \sum_{k=8}^{\infty} e^{-\sqrt{k}} < 1. \tag{10}$$

From the obvious relations

$$\sum_{k=8}^{\infty} e^{-\sqrt{k}} < \int_{7}^{\infty} e^{-\sqrt{k}}\, dk = 2\int_{\sqrt{7}}^{\infty} t\, e^{-t}\, dt = (2 + 2\sqrt{7})e^{-\sqrt{7}} = 0.517347\ldots < 0.52, \tag{11}$$

it follows that (10) will be proved if we establish that

$$\left(\frac{2}{\tau(4)}\right)^{\sigma(4)} + \left(\frac{2}{\tau(6)}\right)^{\sigma(6)} + 0.52 < 1. \tag{12}$$

But the left hand-side of (12) after computation yields $0.578772\ldots$. Therefore (12) is proved.

Hence, Theorem 1 is proved. $\square$

As a Corollary from Lemma 9 we obtain the second main result of the paper

Theorem 2. *For any integer $n \geq 2$ the formula*

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{\psi(k)} \right\rfloor \tag{13}$$

is valid.

Proof. It uses the same ideas as the proof of Theorem 1. Let $n = 2, 3, 4, 5, 6, 7$. Then the direct check shows that (13) is true. Let $n \geq 8$ be an integer. Since for prime k $\frac{2}{\tau(k)} = 1$, we obtain

$$\sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{\psi(k)} = \pi(n) + \left(\frac{2}{\tau(4)}\right)^{\psi(4)} + \left(\frac{2}{\tau(6)}\right)^{\psi(6)} + \sum_{\substack{8 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{\psi(k)}. \tag{14}$$

Since (5) is true, (13) follows from (14) after establishing the inequality

$$\left(\frac{2}{\tau(4)}\right)^{\psi(4)} + \left(\frac{2}{\tau(6)}\right)^{\psi(6)} + \sum_{\substack{8 \leq k \leq n \\ k-\text{composite}}} e^{-\sqrt{k}} < 1. \tag{15}$$

Instead of (15) it is enough to prove that the inequality

$$\left(\frac{2}{\tau(4)}\right)^{\psi(4)} + \left(\frac{2}{\tau(6)}\right)^{\psi(6)} + \sum_{k=8}^{\infty} e^{-\sqrt{k}} < 1 \tag{16}$$

holds. But (16) is certainly true if we establish the validity of the inequality:

$$\left(\frac{2}{\tau(4)}\right)^{\psi(4)} + \left(\frac{2}{\tau(6)}\right)^{\psi(6)} + \int_{7}^{\infty} e^{-\sqrt{k}}\, dk < 1,$$

i.e. from (11), the validity of the inequality

$$\left(\frac{2}{\tau(4)}\right)^{\psi(4)} + \left(\frac{2}{\tau(6)}\right)^{\psi(6)} + 0.52 < 1.$$

The direct computation shows that the left hand-side of the above inequality equals to $0.608036\ldots$.

Therefore, Theorem 2 is proved. □

Let θ be either of the functions σ and ψ. Then Theorem 1 and Theorem 2 may be rewritten in the following form.

Theorem 3. *For any integer $n \geq 2$ the formula*

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{\theta(k)} \right\rfloor \tag{17}$$

is valid.

Since for integer $k \geq 1$ we have $\theta(k) > k-1$, then Theorem 3 is corollary from the following result:

Theorem 4. *For any integer $n \geq 2$ the formula*

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{k-1} \right\rfloor \tag{18}$$

is valid.

Proof. Let $n = 2, \ldots, 14$. Then the direct check shows that (18) is valid. Let an integer n satisfy the inequality $n \geq 15$. Since for prime k $\frac{2}{\tau(k)} = 1$, we obtain

$$\begin{aligned}\sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{k-1} =&\pi(n) + \left(\frac{2}{\tau(4)}\right)^{3} + \left(\frac{2}{\tau(6)}\right)^{5} + \left(\frac{2}{\tau(8)}\right)^{7} + \left(\frac{2}{\tau(9)}\right)^{8} + \left(\frac{2}{\tau(10)}\right)^{9} + \\ &+ \left(\frac{2}{\tau(12)}\right)^{11} + \left(\frac{2}{\tau(14)}\right)^{13} + \sum_{\substack{15 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{k-1}.\end{aligned} \tag{19}$$

Because of Lemma 10 (see (6)), we have:

$$\begin{aligned}&\sum_{\substack{15 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{k-1} < \sum_{\substack{k=15 \\ k-\text{composite}}}^{\infty} \left(\frac{2}{\tau(k)}\right)^{k-1} < \sum_{\substack{k=15 \\ k-\text{composite}}}^{\infty} e^{-\frac{k-1}{\sqrt{k}+1}} < \sum_{k=15}^{\infty} e^{-\frac{k-1}{\sqrt{k}+1}} = \\ &= \sum_{k=15}^{\infty} e^{-(\sqrt{k}-1)} = e \sum_{k=15}^{\infty} e^{-\sqrt{k}} < e \int_{14}^{\infty} e^{-\sqrt{k}}\, dk = 2\, e \int_{\sqrt{14}}^{\infty} e^{-k}\, dk = 2\left(1+\sqrt{14}\right) e^{1-\sqrt{14}} = \\ &= 0.61132 < \ldots < 0.612.\end{aligned} \tag{20}$$

On the other hand we have (after calculation):

$$\sum_{\substack{4 \leq k \leq 14 \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{k-1} = 0.376458\ldots < 0.377. \tag{21}$$

Therefore, (20) and (21) yield:

$$\sum_{\substack{4 \leq k \leq n \\ k-\text{composite}}} \left(\frac{2}{\tau(k)}\right)^{k-1} < 0.612 + 0.377 < 1. \tag{22}$$

Now (19) and (22) yield (18) and Theorem 3 is proved. $\square$

As a generalization of all previous results we propose the following general theorem – the forth main result of the paper – which gives an exact formula (depending on an arbitrary arithmetic function with strictly positive values, satisfying certain condition) for the prime counting function π.

Theorem 5. *Let f be an arithmetic function with strictly positive values and let there exist a composite number $T(f) \geq 4$, such that the inequality*

$$\sum_{\substack{4 \leq k \leq T(f)-1 \\ k-composite}} \left(\frac{2}{\tau(k)}\right)^{f(k)} + \sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}} < 1 \tag{23}$$

holds. Then for any integer $n \geq 2$

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{f(k)} \right\rfloor. \tag{24}$$

Remark 1. Further we suppose that $T(f)$ is the minimal composite number satisfying (23).

Remark 2. For $T(f) = 4$, (23) is reduced to the condition

$$\sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}} < 1.$$

Proof of the theorem. Let the integer $n \geq 2$ be arbitrarily chosen. We have

$$\sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{f(k)} = \sum_{\substack{2 \leq k \leq n \\ k \text{ - prime}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} + \sum_{\substack{4 \leq k \leq n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)}, \tag{25}$$

where the second sum in the right hand-side exists if and only if $n \geq 4$.

Since $\frac{2}{\tau(k)} = 1$ for prime k, (25) becomes

$$\sum_{k=2}^{n} \left(\frac{2}{\tau(k)}\right)^{f(k)} = \pi(n) + \sum_{\substack{4 \leq k \leq n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)}. \tag{26}$$

Obviously for $n \leq 3$ (24) is true.

Let $n \geq 4$ and $T(f)$ is the number from condition (23) of Theorem 5 (see also Remark 1 and 2). If $n < T(f)$, then (23) implies

$$\sum_{\substack{4 \leq k \leq n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} < 1.$$

Therefore, (26) yields (24).

Let $n \geq T(f)$. Then:

$$\sum_{\substack{4 \le k \le n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} = \sum_{\substack{4 \le k \le T(f)-1 \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} + \sum_{\substack{T(f) \le k \le n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} <$$

$$< \sum_{\substack{4 \le k \le T(f)-1 \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} + \sum_{\substack{T(f)=k \\ k \text{ - composite}}}^{\infty} \left(\frac{2}{\tau(k)}\right)^{f(k)} < \text{ (because of Lemma 10)}$$

$$< \sum_{\substack{4 \le k \le T(f)-1 \\ k - \text{composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} + \sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}}.$$

From the above relations the inequality

$$\sum_{\substack{4 \le k \le n \\ k \text{ - composite}}} \left(\frac{2}{\tau(k)}\right)^{f(k)} < 1 \tag{27}$$

holds, because of (23).

Now (26) and (27) imply (24) and Theorem 5 is proved. □

In conclusion we note that the present investigation is inspired by the formula:

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{2}{\tau(k)} \right\rfloor .$$

The validity of the above formula is guaranteed by the fact that:

$$\left\lfloor \frac{2}{\tau(k)} \right\rfloor = \begin{cases} 1 & \text{if } k \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

References

[1] Sierpiński, W. *Elementary Number Theory*, 2nd Edition, North Holland, Amsterdam, 1988.

[2] Mitrinović, D., M. Popadić. *Inequalities in Number Theory*. University of Niš, 1978.