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The Fibonacci sequence and the golden ratio in music

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Abstract: This paper presents an original composition based on Fibonacci numbers, to explore the inherent aesthetic appeal of the Fibonacci sequence. It also notes the use of the golden ratio in certain musical works by Debussy and in the proportions of violins created by Stradivarius.
Keywords: Fibonacci sequence, Golden ratio, Musical composition.
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1 Introduction

It is well known that the Fibonacci sequence of numbers and the associated "golden ratio" are manifested in nature and in certain works of art [1]. It is less well known that these numbers also underlie certain musical intervals and compositions.

This paper considers the presence of the Fibonacci sequence in the structure of the octave scale and also notes the use of the golden ratio in instrument design and in certain musical works of composer Claude Debussy. This paper also presents an original composition based on Fibonacci numbers, to explore the inherent aesthetic appeal of this mathematical phenomenon.

2 Music

2.1 Octave scale

An octave is the interval between a note and the next instance of that same note name on the piano. In Fig. 1 an octave interval is from the C on the left to the C on the right of the keyboard. An octave spans 13 notes. For example, an octave starting on C would include C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C. This is called a "chromatic" scale. The interval between two consecutive notes in a chromatic scale is a "semitone" interval. A "whole-tone" interval is twice a semitone interval. The interval between F and G in Fig. 1 is a whole-tone. "Major" and "minor" scales span 8 notes in one octave, with a mixture of semitones and whole-tones.

For example, an octave major scale starting on C would include C, D, E, F, G, A, B, C. On a keyboard there are 8 white keys and 5 black keys. The black keys are grouped in 2 and 3.

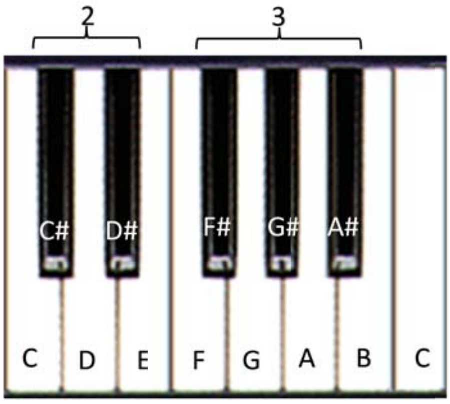

Figure 1.

In the key of C, the notes C, E, G are the basic chord of the key, called the root triad. These are 1, 3, 5 in the scale – Fibonacci numbers. In the octave, the foundational unit of melody and harmony, we see Fibonacci numbers popping up everywhere.

2.2 Instrument design

The greatest of luthiers, Stradivarius, designed his violins around the golden ratio (ϕ). His violins are the most valuable and precious instruments in the string-playing world because of their exquisite tonal and harmonic qualities, [2]. The Stradivarius violin in Fig. 2 reveals how precisely his instruments are determined by the golden ratio, [3]:

$$\frac{a1 + a2}{a2} = \frac{a2}{a1} = \frac{b2}{b1} = \frac{b2}{c2} = \frac{c2}{c1} = \phi$$

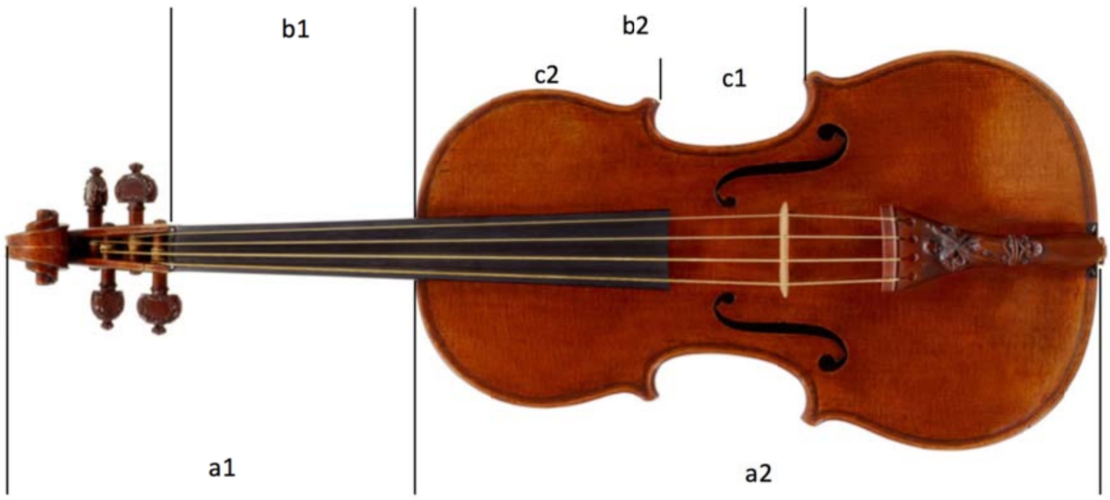

Figure 2. Photo of “Lady Blunt” Stradivarius violin (sold for nearly $16M).
Photo credits: `http://www.bazookaluca.com/2011/07/stradivarius-violins-pizzicato-at-my.html`

2.3 Musical form

Roy Howat in his work, *Debussy in Proportion: A Musical Analysis*, presents his discovery that Debussy’s music “contains intricate proportional systems which can account both for the precise nature of the music’s unorthodox forms and for the difficulty in defining them in more familiar terms”, [4]. These proportional systems are based on the Golden Ratio. For example, Howat notes that the dramatic climax of *Cloches à travers les feuilles* and of ‘Mouvement’ from *Images* (1905) occur exactly on the overall Golden Ratio division of the work, i.e. the climaxes occur when the ratio of the total number of bars to the climax bar gives approximately 1.618.

Howat also postulates that Debussy’s preoccupation with Fibonacci numbers explains some of the unorthodox structure of his compositions. As examples he notes:

- the 21 bars introduction to *Rondes de Printemps*;
- the 34 bars of the first $\frac{3}{8}$ time section of *Jeux*;
- the 34 bars build-up to the triumphant coda of *L’isle joyeuse* and to the recapitulation of *Masques*;
- the 34 bars before the first reprise in *Reflets dans l’eau* and the 55 bars before its climax;
- the 55 bars introduction to the last movement of *La mer*.

3 Fibonacci composition for piano

While not wanting to be compared to the genius of a Debussy, here is a short composition (Fig. 4) based almost entirely on Fibonacci numbers.[1] Each note in the scale of *C major* is numbered (see Fig. 3) and the notes that correspond to Fibonacci numbers are used in the composition. Each time the music changes key, the note that corresponds to 1 in the Fibonacci sequence changes. So in D major, note 1 is a D (while in C major it is a C). The order of notes in the melody is 1, 1, 2, 3, 5, 8 with the occasional addition of 13 and 21.

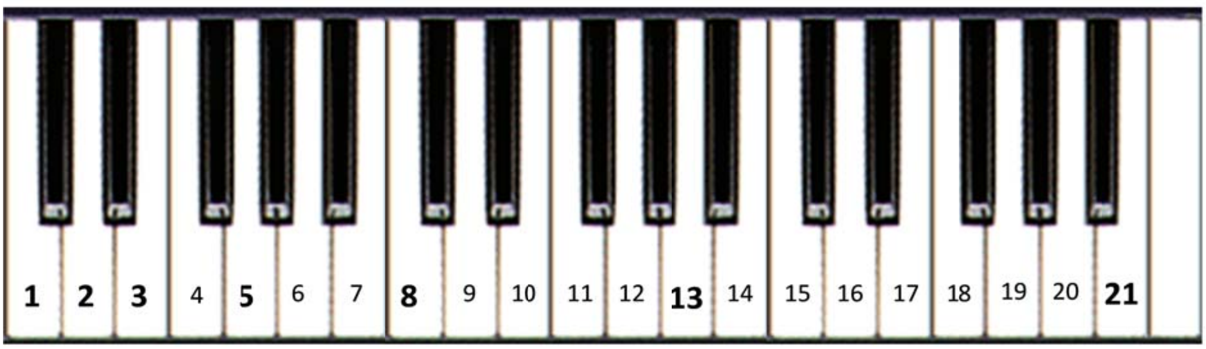

Figure 3.

[1] To hear a world class recording of the work, visit this link and download the file: https://docs.google.com/file/d/0Bwb7y3cfmfoeRV9rZTVObkJKbWs/edit?usp=sharing

Fibonacci Composition

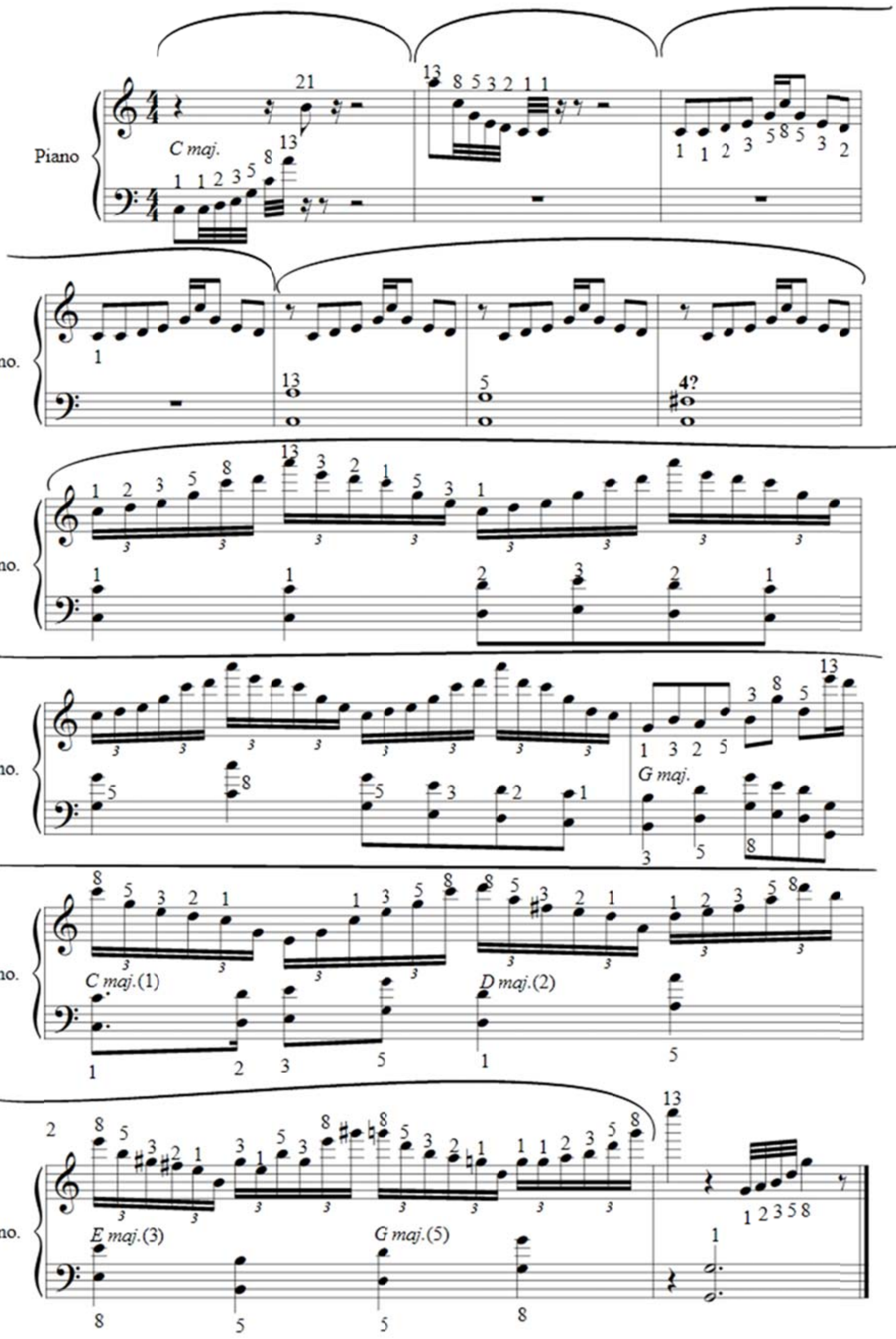

Figure 4.

The work, naturally, is 13 bars long and is structured in phrases of increasing length: 1,1,2,3 and 5 bars. The groups of bars are marked by brackets in the music score (see Fig. 4). After the first 1-bar motif, the second is similar but inverted. Then the theme appears for the first time as a 2-bar phrase, repeated as a 3-bar phrase with added bass notes. A 5-bar developed version of the theme is the final phrase, and a short 1-bar imitation of the first bar motif concludes the work. While this structure is far less subtle than the form of Debussy's compositions, it gives the music a clear feeling of "growth". The theme grows over the course of the work – one can hear in the recording how it sounds "busier" and more developed.

It has been made clear on the score how the notes correspond to Fibonacci numbers. The opening bar is a flourish of notes, using all the Fibonacci numbers in order up to 21. The recurring theme starting in bar 3 uses the notes corresponding to 1, 1, 2, 3, 5, 8, 5, 3, 2, 1. It creates a restful rising and falling tune. In bar 7, a note is used that is not on the Fibonacci sequence. While it makes sense as part of the chord progression in the bass, it sounds less harmonious than the rest of the piece – a "surprise" note. This is an interesting discovery – that a non-Fibonacci note sounds out of place (perhaps even "unnatural") in piece completely full of Fibonacci-notes. In bar 12, I break up the order of the Fibonacci notes for variety and to help with modulating to the new key. Instead of 1, 3, 5, 8, it becomes 3, 1, 5, 3, 8. It has a similar growing sense to it and the notes harmonise well in the progression.

Bar 11 begins a progression of keys. It starts in C major (as indicated on the score) which is number 1 as a Fibonacci note. The music modulates half a bar later to D major, which corresponds to 2 on the Fibonacci sequence. Then the music modulates to E major (3) and finally G major (5). So while the notes themselves fly up and down the Fibonacci notes, the overall progression of keys also follows the sequence: 1, 2, 3, 5. The progression rises well, and has that sense of growth.

From this analysis, it is clear that the sequence 1, 1, 2, 3, 5, 8 has a distinct growing sound to it. In this work, the theme is repeated and developed over the course of the 13 bars, and this gives the piece a feel of continuous rising and falling while evolving. The overall structure of the work is based on groupings of bars into Fibonacci numbers, which gives the sense of expansion and growth of the whole work. The use of only Fibonacci notes works well for harmonious writing. This was surprising, as I thought it would be tough to have variety while only using Fibonacci notes. In summary, it seems that the Fibonacci numbers work naturally together in music too.

4 Conclusion

It is clear that the Fibonacci sequence of numbers and the golden ratio are manifested in music. The numbers are present in the octave, the foundational unit of melody and harmony. Stradivarius used the golden ratio to make the greatest string instruments ever created. Roy Howat's research on Debussy's works shows that the composer used the golden ratio and Fibonacci numbers to structure his music.

The *Fibonacci Composition* reveals the inherent aesthetic appeal of this mathematical phenomenon. Fibonacci numbers harmonise naturally and the exponential growth which the Fibonacci sequence typically defines in nature is made present in music by using Fibonacci notes. Perhaps it is present in other categories of things, such as tastes or smells. It has already been discovered in quantum mechanics and in time, [5].

References

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