On two Diophantine equations

\[ 2A^6 + B^6 = 2C^6 \pm D^3 \]

Susil Kumar Jena

Department of Electronics and Telecommunication Engineering
KIIT University
Bhubaneswar 751024, Odisha, India
e-mail: susil_kumar@yahoo.co.uk

Abstract: We give parametric solutions, and thus show that the two Diophantine equations
\[ 2A^6 + B^6 = 2C^6 \pm D^3 \]
have infinitely many nontrivial and primitive solutions in positive integers \((A, B, C, D)\).

Keywords: Diophantine equation, Diophantine equation \(2A^6 + B^6 = 2C^6 + D^3\), Diophantine equation \(2A^6 + B^6 = 2C^6 - D^3\), Equal sums of higher powers.

AMS Classification: 11D41, 11D72.

1 Introduction

There is extensive study on the Diophantine equation
\[ X_1^6 + X_2^6 + X_3^6 = Y_1^6 + Y_2^6 + Y_3^6, \]  
(1.1)
and many papers (see [1]–[6]) dealing with different aspects of (1.1) have appeared in journals. But, the pair of Diophantine equations
\[ 2X_1^6 + X_2^6 = 2Y_1^6 \pm Y_2^6 \]  
(1.2)
have not yet been investigated. Hence, in this paper, we study two similar Diophantine equations
\[ 2A^6 + B^6 = 2C^6 \pm D^3, \]  
(1.3)
which may raise some hope in dealing with (1.2). Based on an elementary approach, we obtain some parametric solutions for (1.3).
2 Parameterising $2A^6 + B^6 = 2C^6 \pm D^3$

We need the following Lemma for parameterising (1.3).

**Lemma 2.1.** For any real values of $a$ and $b$ there is a polynomial identity

$$(a^2 + ab - b^2)^2 - (a^2 + ab - b^2)(a^2 - ab - b^2) + (a^2 - ab - b^2)^2 = (a^4 + a^2b^2 + b^4). \quad (2.1)$$

**Proof.** Let us expand and simplify the LHS of (2.1).

Using (2.2), (2.3) and (2.4) we get

LHS of (2.1) = $(a^4 + 2a^3b - a^2b^2 - 2ab^3 + b^4) - (a^4 - 3a^3b + b^4) + (a^4 - 2a^3b - a^2b^2 + 2ab^3 + b^4)$

= $(a^4 + 2a^3b - a^2b^2 - 2ab^3 + b^4) + a^4 - 2a^3b - a^2b^2 + 2ab^3 + b^4)$

= $(a^4 + a^2b^2 + b^4) = $ RHS of (2.1).

Hence, the proof is complete. \(\square\)

Now, we have

$$(a^2 + ab - b^2)^3 + (a^2 - ab - b^2)^3 = \{(a^2 + ab - b^2) + (a^2 - ab - b^2)\times \ (a^2 + ab - b^2)^2 - (a^2 + ab - b^2)(a^2 - ab - b^2) + (a^2 - ab - b^2)^2\}

= 2(a^2 - b^2)(a^4 + a^2b^2 + b^4)[by (2.1)] = 2(a^6 - b^6). \quad (2.5)$$

From (2.5) we get

$$2b^6 + (a^2 + ab - b^2)^3 = 2a^6 - (a^2 - ab - b^2)^3. \quad (2.6)$$

In (2.6) take

$$a^2 + ab - b^2 = c^2. \quad (2.7)$$

By (2.6) and (2.7) we get

$$2b^6 + c^6 = 2a^6 - (a^2 - ab - b^2)^3. \quad (2.8)$$

From (2.7) we have

$$a^2 + ab - b^2 - c^2 = 0;$$
$$\Rightarrow a = \{-b \pm \sqrt{b^2 + 4b^2 + 4c^2}\}/2;$$
$$\Rightarrow a = \{-b \pm \sqrt{5b^2 + 4c^2}\}/2. \quad (2.9)$$
In (2.9) take
\[ d^2 = 5b^2 + 4c^2. \]  
(2.10)

By (2.9) and (2.10) we get
\[ a = (-b \pm d)/2. \]  
(2.11)

From (2.10) we get
\[ d^2 - 4c^2 = 5b^2; \Rightarrow (d + 2c)(d - 2c) = 5b^2. \]  
(2.12)

In (2.12) take
\[ b = b_1b_2; (d + 2c) = 5b_1^2; \text{ and } (d - 2c) = b_2^2. \]  
(2.13)

Now, solving for \( d \) and \( c \) we get
\[ d = (5b_1^2 + b_2^2)/2; \]  
(2.14)

and
\[ c = (5b_1^2 - b_2^2)/4. \]  
(2.15)

In (2.11), substituting \( b \) and \( d \) from (2.13) and (2.14) we get
\[ a = (-b_1b_2 \pm (5b_1^2 + b_2^2)/2)/2; \Rightarrow a = (-2b_1b_2 \pm (5b_1^2 + b_2^2))/4. \]  
(2.16)

In (2.8), take \[ a = (5b_1^2 - 2b_1b_2 + b_2^2)/4, \] \[ b = b_1b_2 \] and \[ c = (5b_1^2 - b_2^2)/4 \] from (2.16), (2.13) and (2.15) respectively to get
\[ 2(b_1b_2)^6 + \{(5b_1^2 - b_2^2)/4\}^6 = 2\{(5b_1^2 - 2b_1b_2 + b_2^2)/4\}^6 \]
\[ - \{((5b_1^2 - 2b_1b_2 + b_2^2)/4)^2 - ((5b_1^2 - 2b_1b_2 + b_2^2)/4)b_1b_2 - (b_1b_2)^2\}^3. \]  
(2.17)

Multiplying both the sides of (2.17) by \( 4^6 \), and simplifying, we get
\[ 2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 - 2b_1b_2 + b_2^2)^6 \]
\[ - \{((5b_1^2 - 2b_1b_2 + b_2^2)^2 - 4(5b_1^2 - 2b_1b_2 + b_2^2)b_1b_2 - (4b_1b_2)^2\}^3; \]
\[ \Rightarrow 2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 - 2b_1b_2 + b_2^2)^6 \]
\[ - \{(5b_1^4 - 40b_1^3b_2 + 60b_1^2b_2^2 - 8b_1b_2^3 + b_2^4\}^3. \]  
(2.18)

Similarly in (2.8), take \[ a = (-5b_1^2 - 2b_1b_2 - b_2^2)/4, \] \[ b = b_1b_2 \] and \[ c = (5b_1^2 - b_2^2)/4 \] from (2.16), (2.13) and (2.15) respectively to get
\[ 2(b_1b_2)^6 + \{(-5b_1^2 - b_2^2)/4\}^6 = 2\{(-5b_1^2 - 2b_1b_2 - b_2^2)/4\}^6 \]
\[ - \{((-5b_1^2 - 2b_1b_2 - b_2^2)^2 - ((-5b_1^2 - 2b_1b_2 - b_2^2)/4)b_1b_2 - (b_1b_2)^2\}^3; \]
\[ \Rightarrow 2(b_1b_2)^6 + \{(-5b_1^2 - b_2^2)/4\}^6 = 2\{(-5b_1^2 + 2b_1b_2 + b_2^2)/4\}^6 \]
\[ - \{((-5b_1^2 + 2b_1b_2 + b_2^2)/4)^2 - ((5b_1^2 + 2b_1b_2 + b_2^2)/4)b_1b_2 - (b_1b_2)^2\}^3. \]  
(2.19)
Multiplying both the sides of (2.19) by $4^6$, and simplifying, we get

$$2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 + 2b_1b_2 + b_2^2)^6$$

$$- \{ (5b_1^2 + 2b_1b_2 + b_2^2)^2 \} + 4(5b_1^2 + 2b_1b_2 + b_2^2)b_1b_2 - (4b_1b_2)^2 \}^3;$$

$$\Rightarrow 2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 + 2b_1b_2 + b_2^2)^6$$

$$- (25b_1^2 + 40b_1^3b_2 + 6b_1^2b_2^2 + 8b_1b_2^3 + b_2^3)^3. \quad (2.20)$$

In (2.18), taking $b_2 = (b_1 + 1)$ we get

$$2\{ 4b_1(b_1 + 1) \}^6 + \{ 5b_1^2 - (b_1 + 1)^2 \}^6 = 2\{ 5b_1^2 - 2b_1(b_1 + 1) + (b_1 + 1)^2 \}^6$$

$$- \{ 25b_1^4 - 40b_1^3(b_1 + 1) + 6b_1^2(b_1 + 1)^2 - 8b_1(b_1 + 1)^3 + (b_1 + 1)^4 \}^3;$$

$$\Rightarrow 2(4b_1(b_1 + 1))^6 + (4b_1^2 - 2b_1 - 1)^6 = 2(4b_1^2 + 1)^6$$

$$- (-16b_1^4 - 48b_1^3 - 12b_1^2 - 4b_1 + 1)^3;$$

$$\Rightarrow 2(4b_1(b_1 + 1))^6 + (4b_1^2 - 2b_1 - 1)^6 = 2(4b_1^2 + 1)^6$$

$$+ (16b_1^4 + 48b_1^3 + 12b_1^2 + 4b_1 - 1)^3. \quad (2.21)$$

In (2.21), taking $b_1 = p/q$, and then multiplying both the sides by $q^{12}$ we get

$$2(4p(p + q))^6 + (4p^2 - 2pq - q^2)^6 = 2(4p^2 + q^2)^6$$

$$+ (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4)^3. \quad (2.22)$$

Now, based on (2.22) and (2.20) we have the following two theorems:

**Theorem 2.2.** The Diophantine equation $2A^6 + B^6 = 2C^6 + D^3$ has infinitely many nontrivial and primitive solutions in positive integers $(A, B, C, D) = \{ 4p(p + q), (4p^2 - 2pq - q^2), (4p^2 + q^2), (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4) \}$ where $p, q \in \mathbb{N}$ such that either (i). $p = q = 1$, or (ii). $p > q$, $\gcd(2p, q) = 1$, and $(p + q)$ has prime factors $\alpha_i, i \in \mathbb{N}$ $\equiv 2, or 3(\mod 4)$.

**Proof.** In (2.22), we have already established that

$$2(4p(p + q))^6 + (4p^2 - 2pq - q^2)^6 = 2(4p^2 + q^2)^6$$

$$+ (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4)^3.$$}

When $p = q = 1$, we get $(A, B, C, D) = (8, 1, 5, 79)$ where $\gcd(8, 1, 5, 79) = 1$. The conditions: $p, q \in \mathbb{N}$, and $p > q$, make $(A, B, C, D)$ always positive for infinitely many $(p, q)$ pairs. Since $\gcd(2p, q) = 1$, $q$ is odd; and $(4p^2 + q^2)$ contains prime factors $\beta_j, j \in \mathbb{N} \equiv 1(\mod 4)$. So, $\gcd((4p^2 + q^2), (p + q)) = 1$; and $\gcd((4p^2 + q^2), 4p) = 1$. Thus, we see that $\gcd(A, C) = 1$, which implies that $\gcd(A, B, C, D) = 1$. Thus, under the given conditions, we get infinitely many nontrivial and primitive solutions for $(A, B, C, D)$.

**Example 2.3.**

$$(p, q) = (2, 1) : \quad 2 \times 24^6 + 11^6 = 2 \times 17^6 + 695^3;$$

$$(p, q) = (4, 3) : \quad 2 \times 112^6 + 31^6 = 2 \times 73^6 + 15391^3.$$
Remark 2.4. In Theorem 2.2, if we allow \((p + q)\) to have a prime factor \(\gamma \equiv 1(\text{mod} \ 4)\), then, there is no guarantee that \(\gcd(A, B, C, D)\) will always be 1 as one can easily verify from Table 2.1.

\begin{table}[h]
\centering
\begin{tabular}{lll}
\hline
\(\gamma = 5\), & \((p, q) = (4, 1)\): & \(2 \times 80^6 + 55^6 = 2 \times 65^6 + 7375^3; \) \\
& & \(\gcd(80, 55, 65, 7375) = 5.\) \\
\hline
\(\gamma = 13\), & \((p, q) = (12, 1)\): & \(2 \times 624^6 + 551^6 = 2 \times 577^6 + 416495^3; \) \\
& & \(\gcd(624, 551, 577, 416495) = 1.\) \\
\hline
\end{tabular}
\end{table}

Theorem 2.5. The Diophantine equation \(2A^6 + B^6 = 2C^6 - D^3\) has infinitely many nontrivial and primitive solutions in positive integers

\[(A, B, C, D) = \{4mn, (5m^2 - n^2), (5m^2 + 2mn + n^2), (25m^4 + 40m^3n + 6m^2n^2 + 8mn^3 + n^4)\},\]

where \(m, n \in \mathbb{N}\) such that \(\gcd(5m, n) = 1, 2m > n, \) and one is odd, the other is even.

Proof. We show that

\[
2(4mn)^6 + (5m^2 - n^2)^6 = (5m^2 + 2mn + n^2)^6
- (25m^4 + 40m^3n + 6m^2n^2 + 8mn^3 + n^4)^3 ,
\]

by substituting \(b_1 = m, \) and \(b_2 = n\) in (2.20). The conditions: \(m, n \in \mathbb{N}, \) and \(2m > n,\) make \((A, B, C, D)\) always positive for infinitely many \((m, n)\) pairs. The condition \(\gcd(5m, n) = 1\) tells that both of \(m\) and \(n\) are not even, and 5 is not a factor of \(n.\) Since both of \(m\) and \(n\) are not odd, \(B = (5m^2 - n^2)\) is odd, and \(B\) does not share a common factor with \(A = 4mn.\) Thus, we prove that \(\gcd(A, B, C, D) = 1,\) so that the numerical solutions we get are primitive. \(\square\)

Example 2.6.

\[(m, n) = (2, 1) : \quad 2 \times 8^6 + 19^6 = 2 \times 25^6 - 761^3; \]
\[(m, n) = (3, 2) : \quad 2 \times 24^6 + 41^6 = 2 \times 61^6 - 4609^3. \]

Acknowledgment

I express my gratitude to Dr. A. Samanta, the founder Chancellor of KIIT University, and Sj. L. Pattajoshi for their continued support and encouragement.
References


