A note on a broken Dirichlet convolution

Emil Daniel Schwab\textsuperscript{1} and Barnabás Bede\textsuperscript{2}

\textsuperscript{1} Department of Mathematical Sciences, The University of Texas at El Paso
El Paso, TX 79968, USA
e-mail: eschwab@utep.edu

\textsuperscript{2} Department of Mathematics, DigiPen Institute of Technology
Redmond, WA 98052, USA
e-mail: bbede@digipen.edu

Abstract: The paper deals with a broken Dirichlet convolution $\otimes$ which is based on using the odd divisors of integers. In addition to presenting characterizations of $\otimes$-multiplicative functions we also show an analogue of Menon’s identity:

$$\sum_{\substack{a \pmod{n} \\ (a,n)_{\otimes}=1}} (a-1,n) = \phi_{\otimes}(n)[\tau(n) - \frac{1}{2}\tau_2(n)],$$

where $(a,n)_{\otimes}$ denotes the greatest common odd divisor of $a$ and $n$, $\phi_{\otimes}(n)$ is the number of integers $a \pmod{n}$ such that $(a,n)_{\otimes} = 1$, $\tau(n)$ is the number of divisors of $n$, and $\tau_2(n)$ is the number of even divisors of $n$.

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1 Introduction

An arithmetical function is a complex-valued function whose domain is the set of positive integers $\mathbb{Z}^+$. The Dirichlet convolution $f \ast g$ of two arithmetical function $f$ and $g$ is defined by

$$(f \ast g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right),$$

where the summation is over all the divisors $d$ of $n$ (the term ”divisor” always means ”positive divisor”). The identity element relative to the Dirichlet convolution is the function $\delta$:
\[ \delta(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{otherwise} 
\end{cases} \]

An arithmetical function \( f \) has a convolution inverse if and only if \( f(1) \neq 0 \). The convolution inverse of the zeta function \( \zeta \) (\( \zeta(n) = 1 \) for any \( n \in \mathbb{Z}^+ \)) is the (classical) Möbius function \( \mu \):

\[ \mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\
0 & \text{if } n \text{ has one or more repeated prime factors.} 
\end{cases} \]

There are many fundamental results about algebras of arithmetical functions with a variety of convolutions. The Davison— or \( K \)—convolution ([2], [10, Chapter 4]) \( f \ast_K g \) of two arithmetical functions \( f \) and \( g \) is defined by

\[ (f \ast_K g)(n) = \sum_{d|n} K(n,d)f(d)g\left(\frac{n}{d}\right), \]

where \( K \) is a complex-valued function on the set of all pairs of positive integers \( (n,d) \) with \( d|n \). If \( K \equiv 1 \) then the \( K \)-convolution is the Dirichlet convolution.

In [12] the \( C \)-algebra of extended arithmetical functions is considered as an incidence algebra of a proper Möbius category. If a category \( C \) is decomposition-finite (i.e. \( C \) is a small category in which for any morphism \( \alpha, \alpha \in MorC \), there are only a finite number of pairs \( (\beta, \gamma) \in MorC \times MorC \) such that \( \gamma\beta = \alpha \)) then the \( C \)-convolution \( \tilde{f} \ast \tilde{g} \) of two incidence functions \( \tilde{f} \) and \( \tilde{g} \) (that is two complex-valued functions defined on the set \( MorC \) of all morphisms of \( C \)) is defined by:

\[ (\tilde{f} \ast \tilde{g})(\alpha) = \sum_{\gamma\beta=\alpha} \tilde{f}(\beta)\tilde{g}(\gamma). \]

The incidence function \( \tilde{\delta} \) defined by

\[ \tilde{\delta}(\alpha) = \begin{cases} 
1 & \text{if } \alpha \text{ is an identity morphism} \\
0 & \text{otherwise} 
\end{cases} \]

is the identity element relative to the \( C \)-convolution \( \ast \). A Möbius category (in the sense of Leroux [9, 1]) is a decomposition-finite category in which an incidence function \( \tilde{f} \) has a convolution inverse if and only if \( \tilde{f}(\alpha) \neq 0 \) for any identity morphism \( \alpha \). The Möbius function \( \tilde{\mu} \) of a Möbius category \( C \) is the convolution inverse of the zeta function \( \tilde{\zeta} \) defined by \( \tilde{\zeta}(\alpha) = 1 \) for any morphism \( \alpha \) of \( C \). Some useful characterizations of a Möbius category \( C \) are given in [1, 7, 8, 9]. The set of all incidence functions \( I(C) \) of a Möbius category \( C \) becomes a \( C \)-algebra with the usual pointwise addition and multiplication and the \( C \)-convolution \( \ast \).

The prime example of a Möbius category (with a single object) is the multiplicative monoid of positive integers \( \mathbb{Z}^+ \), the convolution being the Dirichlet convolution and the associated Möbius function being the classical Möbius function. A simple example of a proper Möbius category is the category \( C_\otimes \) with two objects 1 and 2 and with \( Hom_{C_\otimes}(1,1) = 2\mathbb{Z}^+ - 1 \) (the set of odd
positive integers), $\text{Hom}_{C\otimes}(1, 2) = 2\mathbb{Z}^+$ (the set of even positive integers), $\text{Hom}_{C\otimes}(2, 1) = \emptyset$, $\text{Hom}_{C\otimes}(2, 2) = \{id_2\}$, the composition of morphisms being the usual multiplication of integers. In this case, the $C\otimes$-convolution (called the broken Dirichlet convolution in [12]) $\tilde{f} \otimes \tilde{g}$ of two incidence functions $\tilde{f}$ and $\tilde{g}$ is the following one:

$$n \in \mathbb{Z}^+, \quad (\tilde{f} \otimes \tilde{g})(n) = \tilde{f}(n)\tilde{g}(id_2) + \sum_{\substack{u,v \in \mathbb{Z}^+ \setminus 1 \colon u \neq n \atop \text{gcd}(u,v) = 1 \atop u \neq v \atop v < n \atop d \atop d \in 2\mathbb{Z}^+}} \tilde{f}(u)\tilde{g}(v); \quad (\tilde{f} \otimes \tilde{g})(id_2) = \tilde{f}(id_2)\tilde{g}(id_2).$$

In [12] the elements of the incidence algebra $I(C\otimes)$ are called extended arithmetical functions. Now, \( A = \{ \tilde{f} \in I(C\otimes) | \tilde{f}(id_2) = \tilde{f}(1) \} \) is a subalgebra of the incidence algebra $I(C\otimes)$ (see [12, Remark 4.2]). All elements of this subalgebra $A$ are arithmetical functions and the convolution induced in $A$ for arithmetical functions is the following:

$$n \in \mathbb{Z}^+, \quad (f \otimes g)(n) = f(n)g(1) + \sum_{d \mid n, d < n \atop d \in 2\mathbb{Z}^+} f(d)g\left(\frac{n}{d}\right).$$

It is straightforward to see that the above arithmetical functions convolution is a Davison convolution with:

$$K_{\otimes}(n, d) = \begin{cases} 1 & \text{if } d = n \text{ or } d \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the incidence functions $\tilde{\delta}, \tilde{\zeta}, \tilde{\mu} \in I(C\otimes)$ are elements of the subalgebra $A$ and, as arithmetical functions, they coincide with the arithmetical functions $\delta, \zeta$ and $\mu_{\otimes}$ respectively, where (see [12, Proposition 2.1])

$$\mu_{\otimes}(n) = \begin{cases} \mu(n) & \text{if } n \text{ is odd} \\ -1 & \text{if } n = 2^k \ (k > 0) \\ 0 & \text{if } n \text{ is even, } n \neq 2^k. \end{cases}$$

\section{Odd-multiplicative arithmetical functions}

Following Haukkanen [3], an arithmetical function $f$ is $K$-multiplicative (where $K$ is the basic complex-valued function of a Davison convolution) if

(1) $f(1) = 1$;

(2) $(\forall n \in \mathbb{Z}^+), \ f(n)K(n, d) = f(d)f\left(\frac{n}{d}\right)K(n, d), \ \text{for all } d \mid n.$

In the case of a Möbius category $C$ we say that an incidence function $f \in I(C)$ is $C$-multiplicative (see also [11]) if the following conditions hold:

(1) $f(1) = 1$;

(2) $(\forall \alpha \in \text{Mor}C), \ f(\alpha) = f(\beta)f(\gamma), \ \text{for all } (\beta, \gamma) \in \text{Mor}C \times \text{Mor}C \text{ with } \gamma\beta = \alpha.$

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Now, we call an arithmetical function \( f \) odd-multiplicative if

1. \( f(1) = 1 \);
2. \((\forall n \in \mathbb{Z}^+)\), \( f(n) = f(2^{n(2)} \prod_p [f(p)]^{n(p)}) \), where \( n = 2^{n(2)} \prod_p p^{n(p)} \) is the canonical factorization of \( n \).

**Proposition 2.1.** Let \( f \) be an arithmetical function. The following statements are equivalent:

(i) \( f \) is odd-multiplicative;

(ii) \( f \) is \( C_\otimes \)-multiplicative;

(iii) \( f \) is \( K_\otimes \)-multiplicative.

**Proof.** (i) \( \Rightarrow \) (ii). Let \( n = 2^{n(2)} \prod_p p^{n(p)} \) be the canonical factorization of \( n \) and let \( n = vu \) the product of two positive integers \( u \) and \( v \) such that \( u \) is odd. If \( u = 2^{u(2)} \prod_p p^{u(p)} \) and \( v = 2^{v(2)} \prod_p p^{v(p)} \) are the canonical factorizations of \( u \) and \( v \) respectively then \( u(2) = 0 \), \( u(p) \leq n(p) \) and \( v(2) = n(2) \), \( v(p) = n(p) - u(p) \). It follows:

\[
f(n) = f(2^{n(2)} \prod_p [f(p)]^{n(p)}) = f(2^{u(2)} \prod_p [f(p)]^{u(p)} f(2^{v(2)} \prod_p [f(p)]^{v(p)}) = f(u) f(v).
\]

(ii) \( \Rightarrow \) (iii). If \( d \) is an odd divisor of \( n \) then \( n = \frac{n}{d} d \) is a factorization of the morphism \( n \) in \( C_\otimes \). Therefore \( f(n) = f(d) f(\frac{n}{d}) \). Since \( K_\otimes(n,d) = 0 \) if \( d \) is even, it follows:

\[
f(n) K_\otimes(n,d) = f(d) f(\frac{n}{d}) K_\otimes(n,d) \quad \text{for all } d | n.
\]

(iii) \( \Rightarrow \) (i). Let \( n = 2^{n(2)} \prod_p p^{n(p)} \) be the canonical factorization of \( n \). Since \( \prod_p p^{n(p)} \) is an odd divisor of \( n \) it follows:

\[
f(n) = f(2^{n(2)} f(\prod_p p^{n(p)}).
\]

It remains to be shown that \( f(\prod_p p^{n(p)}) = \prod_p [f(p)]^{n(p)} \) which immediately follows by induction.

**Proposition 2.2.** Let \( f \) be an arithmetical function such that \( f(1) \neq 0 \). The following statements are equivalent:

(i) \( f \) is odd-multiplicative;

(ii) \( f(g \otimes h) = f g \otimes f h \) for any two arithmetical functions \( g \) and \( h \);

(iii) \( f(g \otimes g) = f g \otimes f g \) for any arithmetical function \( g \);

(iv) \( f \tau_\otimes = f \otimes f \), where

\[
\tau_\otimes(n) = \begin{cases} 
\tau(n) & \text{if } n \text{ is odd} \\
1 + \tau(m) & \text{if } n = 2^k m, \ k > 0, \text{ and } m \text{ is odd}
\end{cases}
\]

(\( \tau(n) \) is the number of divisors of \( n \)).
Proof. (i) $\Rightarrow$ (ii).

\[(fg \otimes fh)(n) = f(n)g(n)h(1) + \sum_{d|m, d<n, d \in \mathbb{Z}^+} f(d)g(d)f\left(\frac{n}{d}\right)h\left(\frac{n}{d}\right) =
\]
\[= f(n)[g(n)h(1) + \sum_{d|m, d<n, d \in \mathbb{Z}^+} g(d)h\left(\frac{n}{d}\right)] = [f(g \otimes h)](n).
\]

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (iv). It is straightforward to check that $\zeta \otimes \zeta = \tau_\otimes$. When we put $g = \zeta$ in (iii) we obtain (iv).

(iv) $\Rightarrow$ (i). Since $f(1) = f(1)\tau_\otimes(1) = f(1)f(1)$ and $f(1) \neq 0$, it follows $f(1) = 1$. Now, let $n = 2^{n(2)}\prod_p p^{n(p)}$ be the canonical factorization of $n$. We shall prove by induction on $s = n(2) + \sum_p n(p)$ that

\[f(n) = f(2^{n(2)})\prod_p [f(p)]^{n(p)}.
\]

If $s = 1$ then obviously the equality holds. The equality holds also if $n = 2^k$. So, we assume that $s > 1$ and in the same time that $\tau_\otimes(n) > 2$. We have

\[f(n)\tau_\otimes(n) = 2f(n) + \sum_{d|m, d \neq 1, n, d \in \mathbb{Z}^+} f(d)f\left(\frac{n}{d}\right).
\]

Since $d|n$ and $d \neq 1, n$ it follows, by the hypothesis of induction, that

\[f(d)f\left(\frac{n}{d}\right) = f(2^{d(2)})\prod_p [f(p)]^{d(p)}f(2^{\frac{\phi}{2}(2)})\prod_p [f(p)]^{\frac{\phi}{2}(p)} = f(2^{n(2)})\prod_p [f(p)]^{n(p)}.
\]

Taking into account that $\zeta \otimes \zeta = \tau_\otimes$, we have

\[\sum_{d|m, d \neq 1, n, d \in \mathbb{Z}^+} f(d)f\left(\frac{n}{d}\right) = (\tau_\otimes(n) - 2)f(2^{n(2)})\prod_p [f(p)]^{n(p)},
\]

and therefore

\[f(n) = f(2^{n(2)})\prod_p [f(p)]^{n(p)}.
\]

\[\square
\]

An arithmetical function $f$ is called multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. If $f$ is multiplicative and $f(1) \neq 0$ (i.e. $f$ is not identically zero) then $f(1) = 1$ and $f^{-1}(1) = 1$. Here and in the next Proposition, $f^{-1}$ (for $f^{-1}$) means the inverse of $f$ ($g$, $fg$) relative to the convolution $\otimes$. Note that $C_\otimes$ being a Möbius category, $f(1) \neq 0$ assures the existence of the convolution inverse $f^{-1}$.

**Proposition 2.3.** Let $f$ be a multiplicative arithmetical function such that $f(1) \neq 0$. The following statements are equivalent:

\[\begin{align*}
\end{align*}\]

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(i) \( f \) is odd-multiplicative;

(ii) \( fg^{-1} = (fg)^{-1} \) for any arithmetical function \( g \) with \( g(1) \neq 0; \)

(iii) \( f \mu = f^{-1}; \)

(iv) \( f^{-1}(p^m) = 0 \) for any odd prime \( p \) and any \( m > 1. \)

**Proof.** (i) \( \Rightarrow \) (ii). \( \delta = f \delta = (fg \otimes g^{-1}) = f g \otimes f g^{-1} \) and \( fg^{-1} \otimes f g = f(g^{-1} \otimes g) = f \delta = \delta. \)

(ii) \( \Rightarrow \) (iii). \( f \mu = f \zeta^{-1} = (f \zeta)^{-1} = f^{-1}. \)

(iii) \( \Rightarrow \) (iv). \( f^{-1}(p^m) = f(p^m)\mu\mu(p^m) = f(p^m)\mu(p^m) = 0 \) if \( m > 1. \)

(iv) \( \Rightarrow \) (i). Let \( n = 2^{n(2)} \prod_p p^{n(p)} \) be the canonical factorization of \( n. \) Since \( f \) is multiplicative it follows:

\[
f(n) = f(2^{n(2)} \prod_p p^{n(p)}).
\]

Now, \( 0 = (f \otimes f^{-1})(p^m) = f(p^m) + f(p^{m-1})f^{-1}(p) \) for any odd prime \( p \) and \( m \geq 1. \) Thus, \( f^{-1}(p) = -f(p) \) and \( f(p^m) = f(p^{m-1})f(p). \) Therefore,

\[
f(n) = f(2^{n(2)} \prod_p [f(p)]^{n(p)}).
\]

\( \square \)

3 The analogue of Menon’s identity

As a matter of course, the Dirichlet convolution leads us to the divisibility relation on \( \mathbb{Z}^+ \) and the convolution \( \otimes \) leads us to an ”odd-divisibility” relation \( |_\otimes \) defined by

\[
m|_\otimes n \text{ if and only if } m \text{ is odd and } m|n.
\]

We denote the greatest common odd divisor of \( m \) and \( n \) by \( (m, n)_\otimes \) and let \( \phi_\otimes(n) \) be the number of integers \( a \mod n \) such that \( (a, n)_\otimes = 1. \)

**Lemma 3.1.** We have:

1. \( (a, n)_\otimes = (a + n, 2n)_\otimes; \)
2. \( \phi_\otimes(2n) = 2\phi_\otimes(n); \)

**Proof.** (1). If \( (a, n)_\otimes = d \) then \( d \) is odd, \( d|a \) and \( d|n. \) It follows that \( d|a + n \) and \( d|2n. \) Therefore, \( d|(a + n, 2n)_\otimes. \) If \( d' \) is an odd integer such that \( d'|a + n \) and \( d'|2n \) then \( d'|n \) and \( d'|a. \) It follows \( (a + n, 2n)_\otimes|d, \) and in conclusion, \( (a, n)_\otimes = (a + n, 2n)_\otimes. \)

(2) follows immediately from (1).

By induction on \( k, \) using Lemma 3.1.(2), we obtain the following result.

**Proposition 3.1.** Let \( n = 2^k m \) be the factorization of \( n \) such that \( m \) is odd. Then

\[
\phi_\otimes(n) = 2^k \phi(m),
\]

where \( \phi \) is Euler’s totient function.
Corollary 3.1. We have
\[
\phi_{\otimes}(n) = \begin{cases} 
\phi(n) & \text{if } n \text{ is odd} \\
2\phi(n) & \text{if } n \text{ is even}.
\end{cases}
\]

Corollary 3.2. The arithmetical function \(\phi_{\otimes}\) is multiplicative.

In the theory of arithmetical functions a well known and elegant result is Menon’s identity ([6]):
\[
\sum_{a \pmod{n}} (a - 1, n) = \phi(n)\tau(n).
\]

In this section, using Menon’s generalized identity established by Haukkanen [5], we evaluate the sum
\[
\sum_{a \pmod{n}} (a - 1, n)
\]
which obviously becomes the above expression in the case if \(n\) is odd.

In [4], Haukkanen introduced the concept of a generalized divisibility relation (of type \(f = \{f_p : p \text{ is prime}\}\)) satisfying certain conditions (see also [5, Section 2]). For such a generalized divisibility relation \(\lambda\), \(f_p\) are functions from \(\mathbb{Z}^+\) to \(\mathbb{Z}^+ \cup \{0\}\) defined by: \(f_p(a)\) is the smallest integer \(i \in \{1, 2, \cdots a\}\) such that \(p^i \mid p^a\) if such \(i\) exists, and \(f_p(a) = 0\) otherwise. Now, \((m, n)_\lambda\) denotes the greatest element among the divisors \(d\) of \(m\) satisfying \(d \nmid n\) and \(\phi_{\lambda}(n)\) is the number of integers \(a \pmod{n}\) such that \((a, n)_\lambda = 1\) (see [5, Section 3]). In [5, Theorem 4.1], Haukkanen established Menon’s generalized identity. In particular (see [5, (4.4)],
\[
\sum_{d \mid n} \phi_{\lambda}(n) = \sum_{d \mid n} \phi(d)n_d = \phi_{\lambda}(n)[\tau(n) - \frac{1}{2}\tau_2(n)],
\]

where \(n_d = \prod_{p \mid d} p^{\nu(p)}\).

Proposition 3.2. We have
\[
\sum_{a \pmod{n}} (a - 1, n) = \phi_{\lambda}(n)[\tau(n) - \frac{1}{2}\tau_2(n)],
\]

where \(\tau_2(n)\) is the number of even divisors of \(n\).

Proof. It is straightforward to check that the relation \(\otimes\) is a Haukkanen’s generalized divisibility relation of type \(f = (0_2, \zeta, \zeta, \cdots)\), where \(0_2(a) = 0\) for any positive integer \(a\). Since
\[
\sum_{d \mid n} \phi_{\otimes}(n_d) = \sum_{d \mid n} \phi(d)n_d = \sum_{d \mid n} 1 = \text{the number of odd divisors of } n,
\]
and

\[
\sum_{\substack{d \mid n \\ d \in 2\mathbb{Z}^+}} \frac{\phi(d)n_d}{d \phi_\otimes(n_d)^{n_d=2^n(d)m_d}} = \sum_{\substack{d \mid n \\ d \in 2\mathbb{Z}^+}} \frac{\phi(d)n_d}{d 2^n(d) \phi(m_d)} = \\
= \sum_{\substack{d \mid n \\ d \in 2\mathbb{Z}^+}} \frac{n_d d \prod_{p | d} \left(1 - \frac{1}{p}\right)}{d 2^n(d) m_d \prod_{p | d; p \neq 2} \left(1 - \frac{1}{p}\right)} = \sum_{\substack{d \mid n \\ d \in 2\mathbb{Z}^+}} \left(1 - \frac{1}{2}\right) = \frac{1}{2} \tau_2(n),
\]

it follows that

\[
\sum_{\substack{a \ (mod \ n) \\ (a, n) \otimes = 1}} (a - 1, n) = \phi_\otimes(n) \sum_{d | n} \frac{\phi(d)n_d}{d \phi_\otimes(n_d)} = \\
\phi_\otimes(n)\left[ \sum_{d | n, d \in 2\mathbb{Z}^+ - 1} \frac{\phi(d)n_d}{d \phi_\otimes(n_d)} + \sum_{d | n, d \in 2\mathbb{Z}^+} \frac{\phi(d)n_d}{d \phi_\otimes(n_d)} \right] = \\
\phi_\otimes(n)\left[ \tau(n) - \frac{1}{2} \tau_2(n) \right].
\]

\[\Box\]

References


