Generalized Euler–Seidel method for second order recurrence relations

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Abstract: We obtain identities for the generalized second order recurrence relation by using the generalized Euler–Seidel matrix with parameters $x, y$. As a consequence, we give some properties and generating functions of well-known special integer sequences.

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1 Introduction

Let $(a_n)$ be a sequence. In [2], the Euler–Seidel matrix associated with this sequence is determined recursively by the formula

\[
\begin{align*}
a_n^0 &= a_n \quad (n \geq 0) \\
a_n^k &= a_{n-1}^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0, \ k \geq 1).
\end{align*}
\]

From relation (1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

\[
a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_0^k.
\]
\[ a^0_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^k_0. \] (3)

Also any entry \( a^k_n \) can be written in terms of the initial sequence as:

\[ a^k_n = \sum_{i=0}^{k} \binom{k}{i} a^0_{n+i}. \] (4)

**Proposition 1. (Euler) [4]** Let

\[ a(t) = \sum_{n=0}^{\infty} a^0_n t^n \]

be the generating function of the initial sequence \( (a^0_n) \). Then the generating function of the sequence \( (a^n_0) \) is

\[ \overline{a}(t) = \sum_{n=0}^{\infty} a^n_0 t^n = \frac{1}{1-t} a \left( \frac{t}{1-t} \right). \] (5)

**Proposition 2. (Seidel) [9]** Let

\[ A(t) = \sum_{n=0}^{\infty} a^0_n \frac{t^n}{n!} \]

be the exponential generating function of the initial sequence \( (a^0_n) \). Then the exponential generating function of the sequence \( (a^n_0) \) is

\[ \overline{A}(t) = \sum_{n=0}^{\infty} a^n_0 \frac{t^n}{n!} = e^t A(t). \] (6)

In fact, it is possible to state a more general result than (6). The following equation gives relation between exponential generating function of columns (or rows) with the exponential generating function of the initial sequence (see [2]).

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a^k_n \frac{u^k t^n}{k! n!} = e^{u A (t + u)}. \] (7)

In [7] there are applications of Euler–Seidel matrix for hyperharmonic and \( r \)-Stirling numbers. Also authors introduced "symmetric infinite matrix" and give some applications in [3].

In [5] the generalized second order recurrence sequence \( \{W_n (a, b; p, q)\} \) is defined as for \( n \geq 0 \)

\[ W_{n+2} = pW_{n+1} - qW_n \] (8)

with initial conditions

\[ W_0 = a, \quad W_1 = b, \]

where \( p^2 - 4q > 0 \). Let the roots of the equation \( t^2 - pt + q = 0 \) be \( \alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \) and \( \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \). Then \( W_n \) can be written in the form
where $A = \frac{b-a\beta}{a-\beta}$ and $B = \frac{aa-b}{a-\beta}$. The following generating functions of \{W_n\} are given in [6, 8] as:

$$
\sum_{n=0}^{\infty} W_n t^n = \frac{a + (b - pa) t}{1 - pt + qt^2}
$$

and

$$
\sum_{n=0}^{\infty} W_n n^n/n! = Ae^{\alpha t} + Be^{\beta t}.
$$

Mezö gave the generating functions of the general second-order recurrence relations in [8]. Here, we get some relation and generating functions of the general second-order recurrence relations by using generalized Euler–Seidel matrices.

The special cases of \{W_n (a, b; p, q)\} give Fibonacci numbers $F_n$ (Oeis A000045), Lucas numbers $L_n$ (Oeis A000032), Pell numbers (or Silver Fibonacci numbers) $P_n$ (Oeis A000129), Pell–Lucas numbers $Q_n$ (Oeis A002203), Jacobsthal numbers $J_n$ (Oeis A001045), Jacobsthal–Lucas numbers $j_n$ (Oeis A014551), Bronze Fibonacci numbers $B_n$ (Oeis A006190), Signed Fibonacci numbers $F_n$ (Oeis A039834), Signed Pell numbers $P_n$ (Oeis A215936).

Also we get the sequences; $D_n$: denominators of continued fraction convergents to $\sqrt{5}$ (Oeis A001076) and $N_n$: numerators of continued fraction convergents to $\sqrt{2}$ (Oeis A001333) as follows:

\[
\begin{align*}
W_n (0, 1; 1, -1) &= F_n, & W_n (2, 1; 1, -1) &= L_n, \\
W_n (0, 1; 2, -1) &= P_n, & W_n (2, 2; 2, -1) &= Q_n, \\
W_n (0, 1; 1, -2) &= J_n, & W_n (2, 1; 1, -2) &= j_n, \\
W_n (0, 1; 3, -1) &= B_n, & W_n (1, 1; -1, -1) &= F_n, \\
W_n (0, 1; -2, -1) &= P_n, & W_n (0, 1; 4, -1) &= D_n, \\
W_n (1, 1; 2, -1) &= N_n.
\end{align*}
\]

2 Generalized Euler–Seidel matrices with two parameters

In this section, we consider the generalized Euler–Seidel matrix, which is given in [1] with parameters $x$, $y$. We obtain the connection between the generating functions of the initial sequence and the first column entries of the generalized Euler–Seidel matrices.

Let us consider a given sequence $(a_n)_{n \geq 0}$. Generalized Euler–Seidel matrix with parameters $x$ and $y$ (see [1]) corresponding to this sequence is recursively defined by the formulae

\[
\begin{align*}
a_n^0 &= a_n \quad (n \geq 0) \\
axk &= xax^{-1} + ya_{n+1}^{-1} \quad (n \geq 0, k \geq 1 \text{ positive integers}).
\end{align*}
\]

where $a_n^k$ represents the $k$-th row and $n$-th column entry and $x$ and $y$ are nonzero real parameters; i.e;
From now on for the sake of simplicity we represent \( a^k_n (x, y) \) with \( a^k_n \).

The following proposition gives the relation between the any entry of the matrix and the initial sequence.

**Proposition 3.** [1] We have

\[
a^k_n = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i a^0_{n+i}.
\]

(13)

**Proof.** By induction on \( n+k \). \( \Box \)

The first row and column can be transformed into each other via the well known binomial inverse pair as follows.

**Corollary 4.**

\[
a^0_n = x^n \sum_{i=0}^{n} \binom{n}{i} \left( \frac{y}{x} \right)^i a^0_{n+i}.
\]

(14)

and

\[
a^0_n = \frac{1}{y^n} \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} a^i_0.
\]

(15)

**Generating Functions.** We give connections between the generating functions of the initial sequences and the first column entries.

**Proposition 5.** The recurrence (12) gives the following relation:

\[
\overline{a}_{x,y}(t) = \frac{1}{1 - xt} a_{x,y} \left( \frac{yt}{1 - xt} \right)
\]

(16)

where

\[
\overline{a}_{x,y}(t) = \sum_{n=0}^{\infty} a^0_n t^n \quad \text{and} \quad a_{x,y}(t) = \sum_{n=0}^{\infty} a^0_n t^n.
\]

**Proof.** Considering (12) we write

\[
\overline{a}_{x,y}(t) = \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r a^0_r \right) t^n.
\]
By changing the order of the above sums and using Newton binomial sums formula we obtain

\[
\overline{a}_{x,y}(t) = \sum_{r=0}^{\infty} \left( \frac{y}{x} \right)^r a^0_r \sum_{n=0}^{\infty} \binom{n+r}{r} (xt)^{n+r}
\]

\[
= \frac{1}{1-xt} \sum_{r=0}^{\infty} a^0_r \left( \frac{yt}{1-xt} \right)^r.
\]

This completes the proof. \qed

Now we give the generalization of the equation (7).

**Proposition 6.** For the \(a_k^n\) entries of the Generalized Euler–Seidel Matrices we have:

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k^n \frac{u^k t^n}{k! n!} = e^{xu} A_{x,y} \left( t + yu \right)
\]

where

\[
A_{x,y}(t) = \sum_{n=0}^{\infty} a^0_n \frac{t^n}{n!}.
\]

**Proof.** Using (13) we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i a^0_{n+i} \right) \frac{u^k t^n}{k! n!} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{x^{k-i} u^{k-i}}{(k-i)!} \sum_{n=0}^{\infty} a^0_{n+i} \frac{t^n}{n!} \frac{(yu)^i}{i!}.
\]

If we write RHS by means of Cauchy product we get:

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k^n \frac{u^k t^n}{k! n!} = \sum_{k=0}^{\infty} \frac{(xu)^k}{k!} \sum_{n=0}^{\infty} \frac{a^0_{n+k} t^n}{n!} \frac{(yu)^k}{k!}.
\]

We can equally well write the last sum in the form \(A_{x,y}(t + yu)\), which completes the proof. \qed

The following corollary also provides the connection between the exponential generating functions of the initial sequence and the first column entries.

**Corollary 7.** \([1]\) The following relation holds:

\[
\overline{A}_{x,y}(t) = e^{xt} A_{x,y}(yt)
\]

where

\[
\overline{A}_{x,y}(t) = \sum_{n=0}^{\infty} a^0_n \frac{t^n}{n!} \quad \text{and} \quad A_{x,y}(t) = \sum_{n=0}^{\infty} a^0_n \frac{t^n}{n!}.
\]

### 3 Applications of generalized Euler–Seidel matrix

In this section, we show that the generalized Euler–Seidel method is useful to obtain some properties of the generalized second order recurrence relation.
Proposition 8.

\[ W_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} (-q)^{k-i} p^i W_{n+i}. \]  
(18)

Proof. By setting \( x = -q \) and \( y = p \) in (12), we obtain

\[ a_n^k = -qa_n^{k-1} + pa_{n+1}^{k-1}. \]  
(19)

For \( a_n^0 = W_n \), \( n \geq 0 \). We can write \( a_n^1 = W_{n+2} \). By induction on \( k \) and using equation (19), we obtain \( a_n^k = W_{n+2k} \). Now considering equation (13) for \( x = -q \) and \( y = p \), we have

\[ a_n^k = \sum_{i=0}^{k} \binom{k}{i} (-q)^{k-i} p^i a_{n+i}^0. \]

Then we obtain

\[ W_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} (-q)^{k-i} p^i W_{n+i}. \]

This completes the proof. \( \square \)

Using (18), we get the following identities of the Fibonacci numbers \( F_n \), Lucas numbers \( L_n \), Pell numbers \( P_n \), Pell–Lucas numbers \( Q_n \), Jacobsthal numbers \( J_n \), Jacobsthal–Lucas numbers \( j_n \), Bronze Fibonacci numbers \( B_n \), Signed Fibonacci numbers \( B_n \), Signed Pell numbers \( P_n \), and also \( D_n \) and \( N_n \) numbers

- \( F_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} F_{n+i} \)
- \( L_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} L_{n+i} \)
- \( P_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 2^i P_{n+i} \)
- \( Q_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 2^i Q_{n+i} \)
- \( J_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 2^{k-i} J_{n+i} \)
- \( j_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 2^{k-i} j_{n+i} \)
- \( B_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 3^i B_{n+i} \)
- \( F_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i F_{n+i} \)
- \( D_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 4^i D_{n+i} \)
- \( N_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} 2^i N_{n+i} \)

Corollary 9.

\[ W_{2n} = \sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} p^i W_i, \]  
(20)

\[ W_n = \frac{1}{p^n} \sum_{i=0}^{n} \binom{n}{i} (q)^{n-i} W_{2i}, \]  
(21)

and

\[ W_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} p^i W_{i+1}, \]  
(22)

\[ W_n = \frac{1}{p^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (q)^{n-i} W_{2i-1}. \]  
(23)
From (20), we obtain some formulas for these well-known sequences by the new method.

\[
F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i, \quad L_{2n} = \sum_{i=0}^{n} \binom{n}{i} L_i,
\]

\[
P_{2n} = \sum_{i=0}^{n} \binom{n}{i} 2^i P_i, \quad Q_{2n} = \sum_{i=0}^{n} \binom{n}{i} 2^i Q_i,
\]

\[
J_{2n} = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} J_i, \quad j_{2n} = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} j_i,
\]

\[
B_{2n} = \sum_{i=0}^{n} \binom{n}{i} 3^i B_i, \quad F_{2n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i F_i,
\]

\[
\mathcal{P}_{2n} = \sum_{i=0}^{n} \binom{n}{i} (-2)^i \mathcal{P}_i, \quad D_{2n} = \sum_{i=0}^{n} \binom{n}{i} 4^i D_i,
\]

\[
N_{2n} = \sum_{i=0}^{n} \binom{n}{i} 2^i N_i.
\]

Here with help of equation (21), we have following identities:

\[
F_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} F_{2i}, \quad L_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} L_{2i},
\]

\[
P_n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} P_{2i}, \quad Q_n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} Q_{2i},
\]

\[
J_n = \sum_{i=0}^{n} \binom{n}{i} (-2)^{n-i} J_{2i}, \quad j_n = \sum_{i=0}^{n} \binom{n}{i} (-2)^{n-i} j_{2i},
\]

\[
B_n = \frac{1}{3^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} B_{2i}, \quad F_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{2i},
\]

\[
\mathcal{P}_n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \mathcal{P}_{2i}, \quad D_n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} D_{2i},
\]

\[
N_n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} N_{2i}.
\]

We show from (22)

\[
F_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} F_{i+1}, \quad L_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} L_{i+1},
\]

\[
P_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 2^i P_{i+1}, \quad Q_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 2^i Q_{i+1},
\]

\[
J_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} J_{i+1}, \quad j_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} j_{i+1},
\]

\[
B_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 3^i B_{i+1}, \quad F_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{i+1},
\]

\[
\mathcal{P}_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} (-2)^i \mathcal{P}_{i+1}, \quad D_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 4^i D_{i+1},
\]

\[
N_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} 2^i N_{i+1}.
\]
The similar results obtained from equation (23):

\[ F_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} F_{2i-1}, \quad L_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} L_{2i-1}, \]

\[ P_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} P_{2i-1}, \quad Q_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} Q_{2i-1}, \]

\[ J_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-2)^{n-i} J_{2i-1}, \quad j_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-2)^{n-i} j_{2i-1}, \]

\[ B_n = \frac{1}{3^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} B_{2i-1}, \quad F_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{1-i} F_{2i-1}, \]

\[ P_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{i-1} P_{2i-1}, \quad D_n = \frac{1}{4^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} D_{2i-1}, \]

\[ N_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} N_{2i-1}. \]

4 Some results on generating functions

4.1 Results on ordinary generating functions

**Proposition 10.** Generating function of the even \( W_n \) numbers is

\[ \sum_{n=0}^{\infty} W_{2n} t^n = \frac{a (1 + qt) + (b - pa) pt}{(1 + qt)^2 - p^2 t}. \]  

(24)

**Proof.** Firstly we realize that by setting \( a_0^n = W_n \) in GES we get \( a_0^n = W_{2n} \) (see Eq. (19). Here by considering (16) we have

\[ \overline{a_{-q, p}}(t) = \sum_{n=0}^{\infty} W_{2n} t^n = \frac{1}{1 + qt} a_{-q, p} \left( \frac{pt}{1 + qt} \right). \]

Also we know from equation (10)

\[ a_{-q, p}(t) = \sum_{n=0}^{\infty} W_n t^n = \frac{a + (b - pa) t}{1 - pt + qt^2} \]

which completes the proof. \( \square \)

Using (24), we obtain the generating functions of the Fibonacci numbers \( F_n \), Lucas numbers \( L_n \), Pell numbers \( P_n \), Pell–Lucas numbers \( Q_n \), Jacobsthal numbers \( J_n \), Jacobsthal–Lucas numbers \( j_n \), Bronze Fibonacci numbers \( B_n \), Signed Fibonacci numbers \( F_n \), Signed Pell numbers \( P_n \), and also \( D_n \) and \( N_n \) numbers, respectively.

\[ \sum_{n=0}^{\infty} F_{2n} t^n = \frac{t}{1 - 3t + t^2}, \quad \sum_{n=0}^{\infty} L_{2n} t^n = \frac{2-3t}{1 - 3t + t^2}, \]

\[ \sum_{n=0}^{\infty} P_{2n} t^n = \frac{2t}{1 - 6t + t^2}, \quad \sum_{n=0}^{\infty} Q_{2n} t^n = \frac{2-4t}{1 - 6t + t^2}, \]
\[ \sum_{n=0}^{\infty} J_{2n} t^n = \frac{t}{1 - 5t + 4t^2}, \quad \sum_{n=0}^{\infty} j_{2n} t^n = \frac{2 - 5t}{1 - 5t + 4t^2}, \]
\[ \sum_{n=0}^{\infty} B_{2n} t^n = \frac{3t}{1 - 11t + t^2}, \quad \sum_{n=0}^{\infty} F_{2n} t^n = \frac{1 - 3t}{1 - 3t + t^2}, \]
\[ \sum_{n=0}^{\infty} P_{2n} t^n = \frac{-2t}{1 - 6t + t^2}, \quad \sum_{n=0}^{\infty} D_{2n} t^n = \frac{4t}{1 - 18t + t^2}, \]
\[ \sum_{n=0}^{\infty} N_{2n} t^n = \frac{1 - 3t}{1 - 6t + t^2}. \]

**Proposition 11.** Generating function of the odd \( W_n \) numbers is

\[ \sum_{n=0}^{\infty} W_{2n+1} t^n = \frac{(b - pa)(1 + qt) + ap}{(1 + qt)^2 - p^2 t}. \] \( (25) \)

**Proof.** In view of the recurrence (8) we have,

\[ \sum_{n=0}^{\infty} W_{2n+1} t^n = \frac{1}{p} \left( \sum_{n=0}^{\infty} W_{2n+2} t^n + q \sum_{n=0}^{\infty} W_{2n} t^n \right). \]

Employing (24) on the right in the above equation we obtain (25).

From (25), we get the generating functions for odd indexed of these well-known sequences.

\[ \sum_{n=0}^{\infty} F_{2n+1} t^n = \frac{1-t}{1 - 3t + t^2}, \quad \sum_{n=0}^{\infty} L_{2n+1} t^n = \frac{1+t}{1 - 3t + t^2}, \]
\[ \sum_{n=0}^{\infty} P_{2n+1} t^n = \frac{1-t}{1 - 6t + t^2}, \quad \sum_{n=0}^{\infty} Q_{2n+1} t^n = \frac{2+2t}{1 - 6t + t^2}, \]
\[ \sum_{n=0}^{\infty} J_{2n+1} t^n = \frac{1-2t}{1 - 5t + 4t^2}, \quad \sum_{n=0}^{\infty} j_{2n+1} t^n = \frac{1+2t}{1 - 5t + 4t^2}, \]
\[ \sum_{n=0}^{\infty} B_{2n+1} t^n = \frac{1-t}{1 - 11t + t^2}, \quad \sum_{n=0}^{\infty} F_{2n+1} t^n = \frac{1-2t}{1 - 3t + t^2}, \]
\[ \sum_{n=0}^{\infty} P_{2n+1} t^n = \frac{1-t}{1 - 6t + t^2}, \quad \sum_{n=0}^{\infty} D_{2n+1} t^n = \frac{1-t}{1 - 18t + t^2}, \]
\[ \sum_{n=0}^{\infty} N_{2n+1} t^n = \frac{1+t}{1 - 6t + t^2}. \]

### 4.2 Results on exponential generating functions

**Proposition 12.** Exponential generating function of the \( W_{2n} \) numbers is

\[ \sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} = Ae^{(ap-q)t} + Be^{(bp-q)t}. \] \( (26) \)

**Proof.** For \( a_n^0 = W_n \) in \( GES \) we get \( a_0^n = W_{2n} \) (see Eq. (19)). Using equation (11) we get

\[ A_{-q,p}(t) = \sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} = e^{-qt} \left( Ae^{pt} + Be^{pt} \right), \]

which completes the proof.
From (26)

\[
\sum_{n=0}^{\infty} F_{2n+1} \frac{t^n}{n!} = e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - e^{\left(\frac{3-\sqrt{5}}{2}\right)t},
\]

\[
\sum_{n=0}^{\infty} L_{2n+1} \frac{t^n}{n!} = e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + e^{\left(\frac{3-\sqrt{5}}{2}\right)t},
\]

\[
\sum_{n=0}^{\infty} P_{2n+1} \frac{t^n}{n!} = e^{(3+2\sqrt{2})t} - e^{(3-2\sqrt{2})t} t^{\sqrt{2}},
\]

\[
\sum_{n=0}^{\infty} Q_{2n+1} \frac{t^n}{n!} = e^{(3+2\sqrt{2})t} + e^{(3-2\sqrt{2})t} t^{\sqrt{2}},
\]

\[
\sum_{n=0}^{\infty} J_{2n+1} \frac{t^n}{n!} = e^{(3+2\sqrt{2})t} - e^{(3-2\sqrt{2})t} t^{\sqrt{2}},
\]

\[
\sum_{n=0}^{\infty} J_{2n+1} \frac{t^n}{n!} = e^{4t} - e^{(3-\sqrt{5})t},
\]

\[
\sum_{n=0}^{\infty} B_{2n+1} \frac{t^n}{n!} = e^{(11+3\sqrt{13})t} - e^{(11-3\sqrt{13})t},
\]

\[
\sum_{n=0}^{\infty} B_{2n+1} \frac{t^n}{n!} = e^{(3+2\sqrt{2})t} + e^{(3-2\sqrt{2})t} t^{\sqrt{2}},
\]

Proposition 13. Exponential generating function of the $W_{2n+1}$ numbers is

\[
\sum_{n=0}^{\infty} W_{2n+1} \frac{t^n}{n!} = A \left( p - \frac{q}{\alpha} \right) e^{(\alpha p-Q)t} + B \left( p - \frac{q}{\beta} \right) e^{(\beta p-q)t}.
\]  

(27)

Remark 14. For the sake of simplicity we use the following representation in the proof:

\[
W_e (t) = \sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} \quad \text{and} \quad W_o (t) = \sum_{n=0}^{\infty} W_{2n+1} \frac{t^n}{n!}.
\]

Proof. From equation (8) we have

\[
W_o (t) - b = pW_e (t) - pa - q \int W_o (t) \, dt.
\]

This, combined with (26) to gives

\[
\frac{d}{dt} W_o (t) + qW_o (t) = p \frac{d}{dt} \left( Ae^{(\alpha p-q)t} + Be^{(\beta p-q)t} \right).
\]
Hence we have the following differential equation:

\[ W_o'(t) + qW_o(t) = Ap(\alpha p - q) e^{(\alpha p - q)t} + Bp(\beta p - q) e^{(\beta p - q)t}. \]

The solution of this linear differential equation is:

\[ W_o(t) = A \left( p - \frac{q}{\alpha} \right) e^{(\alpha p - q)t} + B \left( p - \frac{q}{\beta} \right) e^{(\beta p - q)t} + Ke^{-qt}. \]

Considering \( W_o(0) = b \) we calculate the constant \( K \) as

\[ K = b - A \left( p - \frac{q}{\alpha} \right) - B \left( p - \frac{q}{\beta} \right) = 0. \]

Combining these results and after some rearrangement we complete the proof. \( \square \)

Using (26)

\[
\begin{align*}
\sum_{n=0}^{\infty} F_{2n+1} \frac{t^n}{n!} &= \frac{(1+\sqrt{5})e^{\frac{1+\sqrt{5}}{2}t} - (1-\sqrt{5})e^{\frac{1-\sqrt{5}}{2}t}}{2\sqrt{5}}, \\
\sum_{n=0}^{\infty} L_{2n+1} \frac{t^n}{n!} &= \frac{(1+\sqrt{5})e^{\frac{1+\sqrt{5}}{2}t} + (1-\sqrt{5})e^{\frac{1-\sqrt{5}}{2}t}}{2}, \\
\sum_{n=0}^{\infty} P_{2n+1} \frac{t^n}{n!} &= \frac{(1+\sqrt{2})e^{(3+2\sqrt{2})t} - (1-\sqrt{2})e^{(3-2\sqrt{2})t}}{2\sqrt{2}}, \\
\sum_{n=0}^{\infty} Q_{2n+1} \frac{t^n}{n!} &= (1 + \sqrt{2}) e^{(3+2\sqrt{2})t} + (1 - \sqrt{2}) e^{(3-2\sqrt{2})t}, \\
\sum_{n=0}^{\infty} J_{2n+1} \frac{t^n}{n!} &= 2e^{4t} + e^t, \\
\sum_{n=0}^{\infty} J_{2n+1} \frac{t^n}{n!} &= 2e^{4t} - e^t, \\
\sum_{n=0}^{\infty} B_{2n+1} \frac{t^n}{n!} &= \frac{(3+\sqrt{13})e^{\frac{(11+3\sqrt{13})}{2}t} - (3-\sqrt{13})e^{\frac{(11-3\sqrt{13})}{2}t}}{2\sqrt{13}}, \\
\sum_{n=0}^{\infty} F_{2n+1} \frac{t^n}{n!} &= \frac{(\sqrt{3}+1)e^{\frac{(3-\sqrt{7})}{2}t} + (\sqrt{3}-1)e^{\frac{(3+\sqrt{7})}{2}t}}{2\sqrt{3}}, \\
\sum_{n=0}^{\infty} P_{2n+1} \frac{t^n}{n!} &= \frac{(\sqrt{2}-1)e^{(3-2\sqrt{2})t} - (\sqrt{2}+1)e^{(3+2\sqrt{2})t}}{2\sqrt{2}}, \\
\sum_{n=0}^{\infty} D_{2n+1} \frac{t^n}{n!} &= \frac{(2+\sqrt{5})e^{(9+4\sqrt{5})t} - (2-\sqrt{5})e^{(9-4\sqrt{5})t}}{2\sqrt{5}}, \\
\sum_{n=0}^{\infty} N_{2n+1} \frac{t^n}{n!} &= \frac{(1+\sqrt{2})e^{(3+2\sqrt{2})t} + (1-\sqrt{2})e^{(3-2\sqrt{2})t}}{2}.
\end{align*}
\]
References


