Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 4, 39–51
DOI: 10.7546/nntdm.2020.26.4.39-51

Congruences involving alternating sums related to harmonic numbers and binomial coefficients

Laid Elkhiri[1], **Miloud Mihoubi**[2] **and Abdellah Derbal**[3]

[1] Ibn Khaldoun University, EDPNLHM Laboratory of ENS
Old Kouba, Tiaret, Algeria
e-mail: elkhirialjebre@hotmail.fr, laid@univ-tiaret.dz

[2] USTHB, Faculty of Mathematics, RECITS Laboratory
PO Box 32 16111 El-Alia, Bab-Ezzouar, Algiers, Algeria
e-mail: mmihoubi@usthb.dz, miloudmihoubi@gmail.com

[3] ENS, Department of Mathematics, EDPNLHM Laboratory
Old Kouba, Algiers, Algeria
e-mail: abderbal@yahoo.fr

Received: 15 January 2020 **Revised:** 19 September 2020 **Accepted:** 9 November 2020

Abstract: In 2017, Bing He investigated arithmetic properties to obtain various basic congruences modulo a prime for several alternating sums involving harmonic numbers and binomial coefficients. In this paper we study how we can obtain more congruences modulo a power of a prime number p (super congruences) in the ring of p-integer $\mathbb{Z}_p$ involving binomial coefficients and generalized harmonic numbers.

Keywords: Binomial coefficients, Harmonic numbers, Generalized harmonic numbers, Congruences.
2010 Mathematics Subject Classification: 11A07, 11B65, 11B99.

1 Introduction

In this paper we study how we can obtain more interesting congruences modulo a power of a prime number p (super congruences) in the ring of p-integer $\mathbb{Z}_p$, involving binomial coefficients, harmonic numbers and generalized harmonic numbers. Many great mathematicians have been interested to study congruences of sums concerning harmonic nummber and binomial coefficients such the work of Gould [4], Dilsher [2], Prodinger [10], Choi and Srivastava [1], Sun [13], Meštrović and Andji [8], Wang [15] and other, in our work we are interested to find closed expressions for the sums of the form

$$\sum_{k=1}^{q} (-1)^{k+1} k^l \binom{n}{k} H_{k,m},$$

and congruences of the form

$$\sum_{k=1}^{\frac{p-1}{2}} (-1)^{k+1} k^l \binom{p-1}{k} H_k \pmod{p^3}, \quad \sum_{k=1}^{\left[\frac{p}{3}\right]} (-1)^{k+1} k^l \binom{p-1}{k} H_k \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} (-1)^{k+1} k^l \binom{np-1}{k} H_{k,m} \,(\operatorname{mod} p^r), \; r \in \{2,3,4\},$$

for some values of l, m, where $H_{n,m}$ to be the n-th generalized harmonic number defined by

$$H_0 = 0, \quad H_{n,m} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}.$$

We denote

$$S(n,q,m) = \sum_{k=1}^{q} \frac{(-1)^k}{k^m} \binom{n}{k-1}.$$

The sequence $(S(n,q,m))$ satisfies

$$\begin{aligned} S(n,q,m-1) &= (n+1) S(n,q,m) - nS(n-1,q,m), & (1)\\ S(n,q,0) &= (-1)^q \frac{q}{n} \binom{n}{q}, & (2)\\ S(n,q,1) &= \frac{1}{n+1}\left((-1)^q \binom{n}{q} - 1\right), & (3)\\ S(n,q,m) &= \frac{1}{n+1}\left(\sum_{j=1}^{n} S(j,q,m-1) - 1\right). & (4) \end{aligned}$$

In 2017, He [5] established the following congruences:

Theorem 1.1 ([5]). *Let $p > 3$ be a prime number. We have*

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^{k+1} \binom{p-1}{k} H_{k,3} &\equiv \frac{-1}{3} p B_{p-3} \pmod{p^2},\\ \sum_{k=0}^{p-1} (-1)^{k+1} \binom{p-1}{k} H_{k,2} &\equiv \frac{1}{3} p^2 B_{p-3} \pmod{p^3}, \end{aligned}$$

where B_n is the n-th Bernoulli number defined by

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n-1} \binom{n}{k} B_k, \quad n \geq 1.$$

Motivated by the above work of He [5], we establish in this paper the following results.

Theorem 1.2. *Let $p>3$ be a prime, n be a positive integer and m,l be nonnegative integers such $m-l\in\{2,3,\cdots,p-2\}$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^{k+1}\left((np-1-k)(k+1)^{l}+k^{l+1}\right)\binom{np-1}{k}H_{k,m}\\ \equiv\;&\left(n-1+\frac{1}{m-l}\right)pB_{p-m+l}\pmod{p^2}.\end{aligned}\tag{5}$$

In particular, for $l=0$, $p>3$ and $m\in\{2,\ldots,p-2\}$ we get

$$\sum_{k=1}^{p-1}(-1)^{k+1}\binom{np-1}{k}H_{k,m}\equiv-\left(n-1+\frac{1}{m}\right)pB_{p-m}\pmod{p^2},\tag{6}$$

and for $l=1$ we get

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^{k+1}k\binom{np-1}{k}H_{k,m}\\ \equiv\;&\begin{cases}\frac{1}{2}\left(n-1+\frac{1}{m}\right)pB_{p-m}\pmod{p^2} & \text{if } m \text{ odd,}\\ -\frac{1}{2}\left(n-1+\frac{1}{m-1}\right)pB_{p+1-m}\pmod{p^2} & \text{if } m \text{ even.}\end{cases}\end{aligned}\tag{7}$$

Theorem 1.3. *Let $p>3$ be a prime and m be a integer with $m\in\left\{1,\ldots,\frac{p-3}{2}\right\}$. We have*

$$\sum_{k=0}^{p-1}(-1)^{k+1}\binom{np-1}{k}H_{k,2m}\equiv\frac{A(m,n)}{2(2m+1)}p^2B_{p-1-2m}\pmod{p^3},\tag{8}$$

where

$$A(m,n)=(m+1)(2m+1)n^2+(m+1)(2m-1)n-2m(2m+1).$$

Remark 1. For $m=3$ and $n=1$ in (6) we obtain the congruence (1.2) given in [5, Theorem 1.2], and, for $m=1$ and $n=1$ in (8) we obtain the congruence (1.1) given in [5].

Theorem 1.4. *Let p be a prime number and n be a positive integer. Then*
1) if $p\geq 5$, we have

$$\begin{aligned}\sum_{k=1}^{\frac{p-1}{2}}(-1)^{k+1}\binom{p-1}{k}H_k\;\equiv\;&\left(q_2-\frac{1}{2}\right)+\left(\frac{3}{2}q_2^2+q_2-\frac{1}{2}\right)p\\ &+\left(\frac{1}{3}q_2^3+\frac{1}{2}q_2^2+q_2-\frac{1}{2}+\frac{7}{24}B_{p-3}\right)p^2\pmod{p^3},\end{aligned}$$

2) if $p\equiv 1\pmod 3$, we have

$$\begin{aligned}&\sum_{k=1}^{\frac{p-1}{3}}(-1)^{k+1}\binom{np-1}{k}H_k\\ \equiv\;&\left(q_3-\frac{1}{3}\right)+\left(\frac{3n-1}{2}q_3^2+\frac{n+1}{2}q_3-\frac{2n-1}{3}+\frac{1}{9}B_{p-2}\left(\frac{1}{3}\right)\right)p\pmod{p^2}\end{aligned}$$

and if $p\equiv 2 \pmod 3$, *we have*

$$\begin{aligned}&\sum_{k=1}^{\frac{p-2}{3}}(-1)^{k+1}\binom{np-1}{k}H_k\\ \equiv\ &\frac{1}{2}q_3-\frac{2}{3}+\left(\frac{3n-1}{4}q_3^2-\frac{n-1}{2}q_3-\frac{4n-1}{3}-\frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)\right)p \pmod{p^2},\end{aligned}$$

where q_a *is the quotient of Fermat base* a *defined for a given prime number* p *by*

$$q_a=q_p(a):=\frac{a^{p-1}-1}{p},\quad a\in\mathbb{Z}-p\mathbb{Z},$$

and $\{B_n(x)\}$ *is the sequence of Bernoulli polynomials defined by*

$$B_n(x)=\sum_{k=0}^{n-1}\binom{n}{k}B_{n-k}x^k,\ \ n\geq 1.$$

The next theorem is a generalization of Theorem 1.1 given [5].

Theorem 1.5. *Let* $p>5$ *be a prime number. We have*

$$\begin{aligned}&\sum_{k=1}^{p-1}(-1)^{k+1}\binom{p-1}{k}H_{k,2}\\ \equiv\ &\left(\frac{B_{2p-4}}{2p-4}-2\frac{B_{p-3}}{p-3}\right)p^2+\left(\frac{B_{2p-4}}{2p-4}-2\frac{B_{p-3}}{p-3}\right)p^3 \pmod{p^4}.\end{aligned}$$

Corollary 1.5.1. *Let* $p\geq 5$ *be a prime number and* n *be a positive integer with* $n\not\equiv 0 \pmod p$. *We have*

$$\sum_{k=0}^{p-1}(-1)^{k+1}\binom{n}{k}H_k\equiv\frac{1}{n}\left(1-\binom{n-1}{p-1}\right) \pmod{p^3}.$$

2 Basic tools to prove the main theorems

In order to prove the main Theorems, we need some auxiliary results given by the following Lemmas.

Lemma 2.1 ([11, Th. 5.1, Cor. 5.1])**.** *Let* $p>3$ *be a prime. Then, for* $m\in\{1,3,\ldots,p-4\}$ *being an odd number, we have*

$$\sum_{k=1}^{p-1}\frac{1}{k^m}\equiv\frac{m(m+1)}{2}\frac{B_{p-2-m}}{p-2-m}p^2 \pmod{p^3}, \tag{9}$$

and for $m\in\{1,2,\ldots,p-2\}$, *we have*

$$\sum_{k=1}^{p-1}\frac{1}{k^m}\equiv\frac{mp}{m+1}B_{p-1-m} \pmod{p^2}. \tag{10}$$

In particular, for $m\in\{1,2,\ldots,p-2\}$, *we have*

$$\sum_{k=1}^{p-1}\frac{1}{k^m}\equiv 0 \pmod p. \tag{11}$$

Lemma 2.2 ([13, Lem. 3.1])**.** *Let n and l be positive integers with $l \leq n-1$, and let $p > n$ be a prime number. Then*

$$\sum_{k=1}^{p-1} \frac{H_{k,l}}{k^{n-l}} \equiv \begin{cases} \frac{(-1)^{l-1}}{n}\binom{n}{l} B_{p-n} \pmod{p}, & \text{if } n \text{ odd,} \\ \\ \frac{pB_{p-1-n}}{2(n+1)}\left(n+(-1)^l \frac{n-2l}{l+1}\binom{n+1}{l}\right) \pmod{p^2}, & \text{if } n \text{ even.} \end{cases} \tag{12}$$

Lemma 2.3 ([11, Th. 5.2])**.** *Let $p > 3$ be a prime number. Then*

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \equiv -2q_2 + pq_2^2 - \frac{2}{3}p^2 q_2^3 - \frac{7}{12}p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}. \tag{13}$$

Lemma 2.4. *Let p be an odd prime number. Then, for $n \in \{1, 2, \ldots, p-2\}$ we have*

$$\sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv B_{p-1-n} \pmod{p} \tag{14}$$

and for $n \in \left\{1, 2, \ldots, \frac{p-3}{2}\right\}$ we have

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} \equiv B_{p-1-2n} \pmod{p}. \tag{15}$$

Proof. If n is even, we set $n = 2m$ and $l = 1$ in relation (12), and if n is odd, we use the fact that $H_{p-k} = H_{p-1} - H_{k-1} \equiv -H_{k-1} \pmod{p}$ to get

$$\sum_{k=1}^{p-1} \frac{H_k}{k^n} = \sum_{k=1}^{p-1} \frac{H_{p-k}}{(p-k)^n} \equiv -\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^n} = -\sum_{k=1}^{p-1} \frac{H_k - \frac{1}{k}}{k^n} = -\sum_{k=1}^{p-1} \frac{H_k}{k^n} + \sum_{k=1}^{p-1} \frac{1}{k^{n+1}},$$

then

$$2\sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{n+1}} \equiv 0 = B_{p-1-n} \pmod{p},$$

because $B_{p-1-n} = 0$ when $p-1-n$ is odd. Similarly, we have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} &= \sum_{k=1}^{p-1} \frac{H_{p-k}^2}{(p-k)^{2n-1}} \\ &\equiv -\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}}\left(H_k - \frac{1}{k}\right)^2 \\ &= -\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} + 2\sum_{k=1}^{p-1} \frac{H_k}{k^{2n}} - H_{p-1,2n} \pmod{p}, \end{aligned}$$

so, we get

$$2\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2n-1}} \equiv 2\sum_{k=1}^{p-1} \frac{H_k}{k^{2n}} - H_{p-1,2n} \equiv 2B_{p-1-2n} \pmod{p},$$

which completes the proof. $\square$

Lemma 2.5. *Let p be a prime number and let $k \in \{1, 2, ..., p\}$. Then, for any integer $n \geq 1$, we have*

$$\binom{np-1}{k-1} \equiv (-1)^{k-1} \left(1 - npH_{k-1} + \frac{n^2p^2}{2} \left(H_{k-1}^2 - H_{k-1,2}\right)\right) \pmod{p^3}. \tag{16}$$

Reducing the modulus in this congruence we get

$$\binom{np-1}{k-1} \equiv (-1)^{k-1} (1 - npH_{k-1}) \pmod{p^2}. \tag{17}$$

Proof. As of the proofs given [8, Lem. 2.3], we have

$$\begin{aligned}
\binom{np-1}{k-1} &= \frac{(np-1)(np-2)\cdots(np-j)\cdots(np-(k-1))}{1.2\cdots j\cdots(k-1)} \\
&= \left(\frac{np}{1}-1\right)\left(\frac{np}{2}-1\right)\cdots\left(\frac{np}{j}-1\right)\cdots\left(\frac{np}{k-1}-1\right) \\
&= (-1)^{k-1}\left(1-\frac{np}{1}\right)\left(1-\frac{np}{2}\right)\cdots\left(1-\frac{np}{j}\right)\cdots\left(1-\frac{np}{k-1}\right) \\
&\equiv (-1)^{k-1}\left(1 - np\sum_{i=1}^{k-1}\frac{1}{i} + n^2p^2 \sum_{1\leq i<j\leq k-1}\frac{1}{ij}\right) \\
&\equiv (-1)^{k-1}\left(1 - npH_{k-1} + \frac{n^2p^2}{2}\left(H_{k-1}^2 - H_{k-1,2}\right)\right) \pmod{p^3}.
\end{aligned}$$

□

Lemma 2.6. *Let*

$$S_{n,p,m} := S(np-1, p-1, m) = \sum_{k=1}^{p-1} \frac{(-1)^k}{k^m}\binom{np-1}{k-1}.$$

Then, if $m \in \{1, 2, ..., p-3\}$, we obtain

$$S_{n,p,m} \equiv 0 \pmod{p} \quad \textit{and} \quad S_{n,p,m} \equiv \left(n - \frac{m}{m+1}\right) pB_{p-1-m} \pmod{p^2}, \tag{18}$$

and, if $m \in \left\{1, 2, ..., \frac{p-3}{2}\right\}$, we obtain

$$S_{n,p,2m-1} \equiv -\frac{T(m,n)}{2(2m+1)} p^2 B_{p-1-2m} \pmod{p^3}, \tag{19}$$

where $T(m,n) = (2m+1)(m-1)n^2 + (2m-1)(m+1)n - 2m(2m-1)$.

Proof. In view of the congruence (17) we have

$$\begin{aligned}
S_{n,p,m} &= \sum_{k=1}^{p-1} (-1)^k \frac{1}{k^m}\binom{np-1}{k-1} \\
&\equiv -\sum_{k=1}^{p-1} \frac{1-npH_{k-1}}{k^m} \\
&= -\sum_{k=1}^{p-1}\frac{1}{k^m} + np\sum_{k=1}^{p-1}\frac{H_k - \frac{1}{k}}{k^m} \\
&= -\sum_{k=1}^{p-1}\frac{1}{k^m} + np\left(\sum_{k=1}^{p-1}\frac{H_k}{k^m} - \sum_{k=1}^{p-1}\frac{1}{k^{m+1}}\right) \pmod{p^2},
\end{aligned}$$

using the relations of congruences (10), (11) and (14) the proof of (18) is complete.

The congruence (9) gives

$$\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \equiv \frac{(2m-1)\,m}{p-2m-1} p^2 B_{p-2m-1} \equiv -m \frac{2m-1}{2m+1} p^2 B_{p-2m-1} \pmod{p^3} \tag{20}$$

and by using the above congruences and the congruences (10), (11), (14), (15) and (16) we get

$$\begin{aligned}
S_{n,p,2m-1} &= \sum_{k=1}^{p-1} (-1)^k \frac{1}{k^{2m-1}} \binom{np-1}{k-1} \\
&\equiv -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \left(1 - npH_{k-1} + \frac{n^2p^2}{2}\left(H_{k-1}^2 - H_{k-1,2}\right)\right) \\
&= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} + np \sum_{k=1}^{p-1} \frac{H_{k-1}}{k^{2m-1}} - \frac{n^2p^2}{2}\left(\sum_{k=1}^{p-1} \frac{H_{k-1}^2}{k^{2m-1}} - \sum_{k=1}^{p-1} \frac{H_{k-1,2}}{k^{2m-1}}\right) \\
&= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} + np \sum_{k=1}^{p-1} \frac{H_k - \frac{1}{k}}{k^{2m-1}} - \frac{n^2p^2}{2}\left(\sum_{k=1}^{p-1} \frac{\left(H_k - \frac{1}{k}\right)^2}{k^{2m-1}} - \sum_{k=1}^{p-1} \frac{H_{k,2} - \frac{1}{k^2}}{k^{2m-1}}\right) \\
&= -\sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} + np \left(\sum_{k=1}^{p-1} \frac{H_k}{k^{2m-1}} - \sum_{k=1}^{p-1} \frac{1}{k^{2m}}\right) \\
&\quad -\frac{n^2p^2}{2}\left(\sum_{k=1}^{p-1} \frac{H_k^2}{k^{2m-1}} - 2\sum_{k=1}^{p-1} \frac{H_k}{k^{2m}} + 2\sum_{k=1}^{p-1} \frac{1}{k^{2m+1}} - \sum_{k=1}^{p-1} \frac{H_{k,2}}{k^{2m-1}}\right) \\
&\equiv m\frac{2m-1}{2m+1} p^2 B_{p-2m-1} + np\left(\frac{1+3m-2m^2}{2\,(2m+1)} pB_{p-2m-1} - \frac{2m}{2m+1} pB_{p-1-2m}\right) \\
&\quad -\frac{n^2p^2}{2}\left(B_{p-2m-1} - 2B_{p-1-2m} + mB_{p-1-2m}\right) \pmod{p^3},
\end{aligned}$$

and this is the congruence (19). □

Lemma 2.7. *Let $p > 3$ be a prime number. Then*

$$\sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{1}{k} \equiv -\frac{p}{6} L_{\frac{p}{3}} B_{p-2}\left(\frac{1}{3}\right) - \frac{3}{2} q_3 + \frac{3}{4} p q_3^2 \pmod{p^2}, \tag{21}$$

where $L_{\frac{p}{3}}$ is Legendre symbol defined by

$$L_{\frac{p}{3}} = \begin{cases} \;\;1 & \text{if } p \equiv 1 \pmod 3, \\ -1 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. We have

$$\sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{1}{p-3k} = \sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{p+3k}{(p-3k)(p+3k)} = \sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{p+3k}{p^2-9k^2} \equiv \frac{-1}{9} \sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{p+3k}{k^2},$$

i.e.

$$\sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{1}{p-3k} \equiv -\frac{p}{9} \sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^{\left[\frac{p}{3}\right]} \frac{1}{k} \pmod{p^2},$$

and by the relation 42 given in [7] and Theorem 3.3 given in [12] we obtain

$$\sum_{k=1}^{\left[\frac{p}{3}\right]}\frac{1}{p-3k}\equiv\frac{1}{2}q_3-\frac{1}{4}pq_3^2 \pmod{p^2} \quad and \quad \sum_{k=1}^{\left[\frac{p}{3}\right]}\frac{1}{k^2}\equiv\frac{1}{2}L_{\frac{p}{3}}B_{p-2}\left(\frac{1}{3}\right) \pmod{p}.$$

So, we may state $-\frac{1}{3}\sum_{k=1}^{\left[\frac{p}{3}\right]}\frac{1}{k}\equiv\frac{p}{18}L_{\frac{p}{3}}B_{p-2}\left(\frac{1}{3}\right)+\frac{1}{2}q_3-\frac{1}{4}pq_3^2 \pmod{p^2}$. □

Theorem 2.8. *For $m,l,n\geq 1$, $q\geq 1$ be integers we have*

$$\begin{aligned}&\sum_{k=0}^{q}(-1)^{k+1}\left((n-k)(k+1)^l+k^{l+1}\right)\binom{n}{k}H_{k,m} \\ =\ &(-1)^{q+1}(n-q)(q+1)^l\binom{n}{q}H_{q,m}-nS(n-1,q,m-l).\end{aligned} \tag{22}$$

In particular, for $l=0,1$ we get

$$\sum_{k=0}^{q}(-1)^{k+1}n\binom{n}{k}H_{k,m}=(-1)^{q+1}(n-q)\binom{n}{q}H_{q,m}-nS(n-1,q,m), \tag{23}$$

$$\begin{aligned}&\sum_{k=0}^{q}(-1)^{k+1}(nk+n-k)\binom{n}{k}H_{k,m}= \\ &\qquad(-1)^{q+1}(n-q)(q+1)\binom{n}{q}H_{q,m}-nS(n-1,q,m-1),\end{aligned}$$

for $q=n$ we get

$$\sum_{k=0}^{n}(-1)^{k+1}\left((n-k)(k+1)^l+k^{l+1}\right)\binom{n}{k}H_{k,m}=-nS(n-1,n,m-l).$$

and for $m=1$ and $l=0$ or 1 we get

$$\begin{aligned}n\sum_{k=0}^{q}(-1)^{k+1}\binom{n}{k}H_k &= (-1)^{q+1}(n-q)\binom{n}{q}\left(H_q+\frac{1}{n}\right)+1 \text{ and} \\ \sum_{k=0}^{q}(-1)^{k+1}k\binom{n}{k}H_k &= (-1)^{q+1}\frac{n-q}{n-1}\binom{n}{q}\left(qH_q+\frac{q}{n-1}-\frac{1}{n}\right)-\frac{1}{n-1}.\end{aligned} \tag{24}$$

Proof. We proceed as follows

$$\begin{aligned}&(-1)^q q^{l+1}H_{q,m}\binom{n}{q} \\ &=\sum_{k=0}^{q}(-1)^k k^{l+1}\binom{n}{k}H_{k,m}-\sum_{k=0}^{q-1}(-1)^k k^{l+1}\binom{n}{k}H_{k,m} \\ &=\sum_{k=0}^{q-1}(-1)^{k+1}(k+1)^{l+1}\binom{n}{k+1}H_{k+1,m}+\sum_{k=0}^{q-1}(-1)^{k+1}k^{l+1}\binom{n}{k}H_{k,m}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{q-1} (-1)^{k+1} (n-k)(k+1)^l \binom{n}{k} \left(H_{k,m} + \frac{1}{(k+1)^m} \right) + \sum_{k=0}^{q-1} (-1)^{k+1} k^{l+1} \binom{n}{k} H_{k,m} \\
&= \sum_{k=0}^{q-1} (-1)^{k+1} (n-k)(k+1)^l \binom{n}{k} H_{k,m} + \sum_{k=0}^{q-1} (-1)^{k+1} \frac{n-k}{(k+1)^{m-l}} \binom{n}{k} \\
&\quad + \sum_{k=0}^{q-1} (-1)^{k+1} k^{l+1} \binom{n}{k} H_{k,m} \\
&= \sum_{k=0}^{q-1} (-1)^{k+1} \left((n-k)(k+1)^l + k^{l+1} \right) \binom{n}{k} H_{k,m} + \sum_{k=0}^{q-1} (-1)^{k+1} \frac{n-k}{(k+1)^{m-l}} \binom{n}{k} \\
&= \sum_{k=0}^{q-1} (-1)^{k+1} \left((n-k)(k+1)^l + k^{l+1} \right) \binom{n}{k} H_{k,m} + \sum_{k=1}^{q} (-1)^{k} \frac{n-k+1}{k^{m-l}} \binom{n}{k-1} \\
&= \sum_{k=0}^{q-1} (-1)^{k+1} \left((n-k)(k+1)^l + k^{l+1} \right) \binom{n}{k} H_{k,m} \\
&\quad + (n+1) S(n,q,m-l) - S(n,q,m-l-1) \\
&= \sum_{k=0}^{q-1} (-1)^{k+1} \left((n-k)(k+1)^l + k^{l+1} \right) \binom{n}{k} H_{k,m} + nS(n-1,q,m-l). \qquad \square
\end{aligned}
$$

Remark 2. In Theorem 2.8, if $q = n$, we get the known identity (see, for example [1, 2, 4]). For $m = 1$ we have

$$
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} H_k = \frac{1}{n} \tag{25}
$$

and for $m = 2$ we have

$$
\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k,2} = -\frac{H_n}{n}. \tag{26}
$$

Also, for $q = n$ and $l = m$ or $m - 1$ in Theorem 2.8 we have

$$
\begin{aligned}
\sum_{k=0}^{n} (-1)^{k+1} \left((n-k)(k+1)^m + k^{m+1} \right) \binom{n}{k} H_{k,m} &= 0, \\
\sum_{k=0}^{n} (-1)^{k+1} \left((n-k)(k+1)^{m-1} + k^m \right) \binom{n}{k} H_{k,m} &= 1.
\end{aligned}
$$

3 Proofs of the main theorems

Proof of Theorem 1.2. For $q = p - 1$ and $n := np - 1$ in the identity (22) we get

$$
\begin{aligned}
&\sum_{k=0}^{p-1} (-1)^{k+1} \left((np-1-k)(k+1)^l + k^{l+1} \right) \binom{np-1}{k} H_{k,m} \\
&= (-1)^p (n-1) p^{l+1} \binom{np-1}{p-1} H_{p-1,m} - (np-1) S(np-2, p-1, m-l),
\end{aligned}
$$

and since $H_{p-1,m} \equiv 0 \pmod{p}$ we obtain

$$\begin{aligned}
&\sum_{k=0}^{p-1}(-1)^{k+1}\left((np-1-k)(k+1)^{l}+k^{l+1}\right)\binom{np-1}{k}H_{k,m}\\
\equiv\;& -(np-1)\,S\,(np-2,p-1,m-l)\pmod{p^2}.
\end{aligned}$$

So, for $q=p-1$ and $n:=np-1$ in the recurrence relation (1) and by using the congruence (18), we get

$$\begin{aligned}
&(np-1)\,S\,(np-2,p-1,m-l)\\
=\;& npS\,(np-1,p-1,m-l)-S\,(np-1,p-1,m-l-1)\\
=\;& npS_{n,p,m-l}-S_{n,p,m-l-1}\\
\equiv\;& -S_{n,p,m-l-1}\\
\equiv\;& -\left(n-1+\frac{1}{m-l}\right)pB_{p-m+l}\pmod{p^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=0}^{p-1}(-1)^{k+1}\left((np-1-k)(k+1)^{l}+k^{l+1}\right)\binom{np-1}{k}H_{k,m}\\
\equiv\;& \left(n-1+\frac{1}{m-l}\right)pB_{p-m+l}\pmod{p^2},
\end{aligned}$$

which is the congruence (5).
For $l=0$ in the congruence (5) we get

$$(np-1)\sum_{k=1}^{p-1}(-1)^{k+1}\binom{np-1}{k}H_{k,m}\equiv\left(n-1+\frac{1}{m}\right)pB_{p-m}\pmod{p^2},\tag{27}$$

and since $\frac{1}{np-1}\equiv -1-np\pmod{p^2}$, the congruence (6) follows.
For $l=1$ in the congruence (5) we get

$$\begin{aligned}
&(np-2)\sum_{k=1}^{p-1}(-1)^{k+1}k\binom{np-1}{k}H_{k,m}+(np-1)\sum_{k=1}^{p-1}(-1)^{k+1}\binom{np-1}{k}H_{k,m}\\
\equiv\;& \left(n-1+\frac{1}{m-1}\right)pB_{p-m+1}\pmod{p^2},
\end{aligned}$$

and by the congruence (27) and the congruence

$$\frac{1}{np-2}\equiv-\frac{1}{2}-\frac{1}{4}np\pmod{p^2},$$

the congruence (7) follows. $\square$

Proof of Theorem 1.3. From the relations (23), (10), (18) and (19) we get

$$\begin{aligned}
&(np-1)\sum_{k=0}^{p-1}(-1)^{k+1}\binom{np-1}{k}H_{k,2m}\\
=&\ -(n-1)\,p\binom{np-1}{p-1}H_{p-1,2m}-(np-1)\,S\left(np-2,p-1,2m\right)\\
\equiv&\ -(n-1)\,pH_{p-1,2m}-(np-1)\,S\left(np-2,p-1,2m\right)\\
=&\ -(n-1)\,pH_{p-1,2m}-npS_{n,p,2m}+S_{n,p,2m-1}\\
=&\ -(n-1)\,\frac{2mp^2}{2m+1}B_{p-1-2m}-np^2\left(n-\frac{2m}{2m+1}\right)B_{p-1-2m}-T\left(m,n\right)\frac{p^2B_{p-1-2m}}{2\left(2m+1\right)}\\
\equiv&\ \left(-n^2+\frac{-4m\left(n-1\right)+4nm-T\left(m,n\right)}{2\left(2m+1\right)}\right)p^2B_{p-1-2m}\pmod{p^3},
\end{aligned}$$

and since

$$\frac{1}{np-1}\equiv-1-np-n^2p^2\pmod{p^3},$$

the proof of theorem is complete. □

Proof of Theorem 1.4. 1) If we let $q=\frac{p-1}{2}$ and $n=p-1$ in relation (24), we have then

$$\sum_{k=1}^{\frac{p-1}{2}}(-1)^{k+1}\binom{p-1}{k}H_k=-\frac{1}{2}\left(-1\right)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}\left(H_{\frac{p-1}{2}}+\frac{1}{p-1}\right)+\frac{1}{p-1},$$

using the known congruence obtained by Morley [9]

$$\binom{p-1}{\frac{p-1}{2}}\equiv\left(-1\right)^{\frac{p-1}{2}}4^{p-1}=\left(-1\right)^{\frac{p-1}{2}}\left(1+pq_2\right)^2\pmod{p^3},$$

and the congruence

$$\frac{1}{p-1}\equiv-1-p-p^2\pmod{p^3},$$

and the relations (13), (24) we have

$$\begin{aligned}
&\sum_{k=1}^{\frac{p-1}{2}}(-1)^{k+1}\binom{p-1}{k}H_k\\
\equiv&\ \frac{-1}{2}\left(1+pq_2\right)^2\left(-2q_2+pq_2^2-\frac{2}{3}p^2q_2^3-\frac{7}{12}p^2B_{p-2}\left(\frac{1}{3}\right)+\frac{1}{p-1}\right)+\frac{1}{p-1}\pmod{p^3},
\end{aligned}$$

the proof is complete.

2) a) For $k-1=\left[\frac{p}{3}\right]$ the relation (17) becomes

$$\binom{np-1}{\left[\frac{p}{3}\right]}\equiv\left(-1\right)^{\left[\frac{p}{3}\right]}\left(1-npH_{\left[\frac{p}{3}\right]}\right)\pmod{p^2},$$

and since $H_{\left[\frac{p}{3}\right]}\equiv-\frac{3}{2}q_3 \pmod{p}$ (see [3]) we obtain

$$\binom{np-1}{\left[\frac{p}{3}\right]}\equiv\left(-1\right)^{\left[\frac{p}{3}\right]}\left(1+\frac{3}{2}npq_3\right)\pmod{p^2}.$$

For $p = 1 \pmod 3$ we use the congruence

$$\frac{1}{np-1} \equiv -1 - np \pmod{p^2},$$

and the relations (21), (24) to obtain

$$\begin{aligned}
&\sum_{k=1}^{\frac{p-1}{3}} (-1)^{k+1} \binom{np-1}{k} H_k \\
= \; & \left(1 + \frac{3}{2} npq_3\right) \frac{\frac{p-1}{3} - (np-1)}{np-1} \left(\frac{-p}{6} B_{p-2}\left(\frac{1}{3}\right) - \frac{3}{2} q_3 + \frac{3}{4} pq_3^2 + \frac{1}{np-1}\right) + \frac{1}{np-1} \\
\equiv \; & \left(q_3 - \frac{1}{3}\right) + \left\{\left(\frac{3}{2}n - \frac{1}{2}\right) q_3^2 + \frac{n+1}{2} q_3 - \frac{2}{3}n + \frac{1}{3} + \frac{1}{9} B_{p-2}\left(\frac{1}{3}\right)\right\} p \pmod{p^2}.
\end{aligned}$$

b) The other congruences can be proved similarly as above. □

Proof of Theorem 1.5. By the relation (26) we get

$$\sum_{k=1}^{p-1} (-1)^{k+1} \binom{p-1}{k} H_{k,2} = \frac{H_{p-1}}{p-1},$$

using the congruence of Remark 5.1 given in [11]

$$H_{p-1} \equiv -\left(\frac{B_{2p-4}}{2p-4} - 2\frac{B_{p-3}}{p-3}\right) p^2 \pmod{p^4},$$

and the congruence

$$\frac{1}{p-1} = -1 - p - p^2 - p^3 \pmod{p^4}, \tag{28}$$

the proof is complete. □

References

[1] Choi, J., & Srivastava, H. M. (2011). Some summation formulas involving harmonic numbers and generalized harmonic numbers, *Mathematical and computer Modelling*, 54, 2220–2234.

[2] Dilcher, K. (1995). Some q-series identities related to divisor factors, *Discrete Math.*, 145, 83–93.

[3] Glaisher, J.W.L. (1900). A general congruence theorem relating to the Bernoullian function, *Proc. London Math. Soc.*, 33, 27–56.

[4] Gould, H. W. (1972). *Combinatorial Identities. A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, W. Va.

[5] He, B. (2017). Some congruences on harmonic numbers and binomial sums, *Period. Math. Hung.*, 74, 67–72.

[6] Koparal, S., & Ömür, N. (2016). Congruences related to central binomial coefficients, harmonic and Lucas numbers, *Turk. J. Math.*, 40, 973–985.

[7] Lehmer, E. (1938). On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. Math.*, 39, 350–360.

[8] Meštrović, R., & Andji, M. (2017). Certain congruence for harmonic numbers, *Mathematica Montisnigri*, 38.

[9] Morley, F. (1895). Note on the congruence $2^{4n} \equiv (-1)^n (2n)!/(n!)^2$, where $2n+1$ is a prime, *Ann. Math.*, 9(1–6), 168–170.

[10] Prodinger, H. (2000). A q-analogue of a formula of Hernandez obtained by inverting a result of Dilcher, *Australas. J. Comb.*, 21, 271–274.

[11] Sun, Z. H. (2000). Congruences concerning Bernoulli numbers and Bernoulli polynomials, *Discrete Appl. Math.*, 105, 193–223.

[12] Sun, Z. H. (2008). Congruences involving Bernoulli and Euler numbers, *J. Number Theory*, 128, 280–312.

[13] Sun, Z. W. (2015). Super congruences motivated by e, *J. Number Theory*, 147, 326–341.

[14] Sun, Z. W., & Zhao, L. L. (2013). Arithmetic theory of Harmonic numbers (II), *Colloq. Math.*, 130(1), 67–78.

[15] Wang, W. S. (2015). Sums of involving the Harmonic numbers and the Binomial coefficients, *American Journal of Computational Mathematics*, 5, 96–105.