Notes on Number Theory and Discrete Mathematics
Vol. 18, 2012, No. 2, 34–41

On the solutions of the equation $x^2 + 19^m = y^n$

Bilge Peker[1] and Selin (İnağ) Çenberci[2]

[1] Elementary Mathematics Education, Ahmet Kelesoglu Education Faculty
Konya Necmettin Erbakan University
e-mail: bilge.peker@yahoo.com

[2] Department of Mathematics, Ahmet Kelesoglu Education Faculty
Konya Necmettin Erbakan University
e-mail: inag_s@hotmail.com

Abstract: In this article, we consider the equation $x^2 + 19^m = y^n$, $n > 2,\ m > 0$. We find the solutions of the title equation for not only $2 \mid m$ but also $2 \nmid m$.
Keywords: Exponential Diophantine equation, Primitive divisors, Quadratic field.
AMS Classification: 11D41, 11D61.

1 Introduction

The Diophantine equation which is so-called generalized Ramanujan-Nagell equation $x^2 + k = y^n,\ k, x, y, n \in \mathbb{Z},\ n > 2$ has been studied extensively. When $n = 3$ the equation clarifies an elliptic curve and it is well known as Mordell's equation. Mordell studied this type of equation in detail in his book [12]. When $n > 3$, the equation clarifies a hyperelliptic curve. This case also has a lot of literature. J.H.E. Cohn [8] solved the equation for 77 values of positive k under 100. We are interested in the case when $k = c^m$, c is a positive integer, $m \in \mathbb{N}$ is unknown. There are lots of studies for different cases of c. For example L. Tao [13] considered the equation for $c = 3$ and 5.

In [6] , Cohn considered the title equation for the case $m = 1$. He proved that the equation $x^2 + 19 = y^n$ has two primitive integer solutions with $n \geq 3$, namely the solutions are $(x, y, k, n) = (18, 7, 0, 3)$, $(22434, 55, 0, 5)$. On the other hand, in [2] Arif and Muriefah gave a theorem for the solutions to the equation $x^2 + q^{2k+1} = y^n$ where $k \geq 0$ and $n \geq 5$ is an odd integer. With these conditions the equation $x^2 + q^{2k+1} = y^n$ has exactly two families of solutions given by $(x, y, k, n) = \big(22434.19^{5M}, 55.19^{2M}, 5M, 5\big), \big(2759646.341^{5M}, 377.341^{2M}, 341M, 5\big)$.

İ. N. Cangül et. al. [5] found all solutions of the equation $x^2 + 11^{2k} = y^n,\ x \geq 1,\ y \geq 1,\ k \in \mathbb{N},\ n \geq 3$. E. Demirpolat et. al. [9] solved the equation $x^2 + 11^{2k+1} = y^n$.

H. Zhu and M. Le [14] give all solutions of some generalized Lebesque- Nagell equations $x^2 + q^m = y^n$, where the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$ is one by using known results and elementary arguments.

The aim of this paper is to study the equation $x^2 + 19^m = y^n$, $n > 2$ and $m > 0$. We treat the equation for m is an even and odd separately. For the following process we need the below Lemma.

Lemma 1 *The equation* $19x^2 + 1 = y^n,\ n > 2$ *has no positive integer solution.*

Proof. Suppose (x, y, n) is a positive integer solution. Since $n > 2$ arguing *modulo* 8 one obtains that if there exist integers x, y such that $19x^2 + 1 = y^n$, then y is an odd and x is an even.

Now for the proof we evaluate two different cases.

(i) The first case is with $n = 4$, the equation $19x^2 + 1 = y^4$. Cohn [7] showed that this equation has no positive integer solution.

(ii) The second case is with $n = p$ is an odd prime, the equation $19x^2 + 1 = y^p$. Now we search this equation whether has a positive integer solution or not. We suppose that $\left(1 - \sqrt{-19}x\right)$ and $\left(1 + \sqrt{-19}x\right)$ two ideals of the ring of integers $\mathbb{Z}\left[\sqrt{-19}\right]$ of the field $F = \mathbb{Q}\left[\sqrt{-19}\right]$ are relatively prime. From the unique factorization of prime ideals in $\mathbb{Z}\left[\sqrt{-19}\right]$, we have $\left(1 - \sqrt{-19}x\right) = \wp^p$ for some ideal $\wp$ of $\mathbb{Z}\left[\sqrt{-19}\right]$. $\wp$ is a principal so there exists integers a, b such that $\wp = \left(\dfrac{a + b\sqrt{-19}}{2}\right)$. Hence

$$1 + \sqrt{-19}x = u\left(\frac{a + b\sqrt{-19}}{2}\right)^p$$

where u is a unit in $\mathbb{Z}\left[\sqrt{-19}\right]$. Since the units in $\mathbb{Z}\left[\sqrt{-19}\right]$ are just ± 1, where $a \neq 0 \mod 19$ and a is an even, then b is an even too since $\wp = \left(\dfrac{a + b\sqrt{-19}}{2}\right)$. So we write $a = 2A, b = 2B$. We get without loss of generality

$$\left(1 + \sqrt{-19}x\right) = \left(A + B\sqrt{-19}\right)^p \tag{1.1}$$

If we take the norm of both sides then we get $1 + 19x^2 = (A^2 + 19B^2)^p$. Since $y^p = 1 + 19x^2$ we have $y = A^2 + 19B^2$. Now comparing the real parts of the equality (1.1), then we obtain

$$1 = A_{k=0}^{\frac{p-1}{2}} \left(\begin{array}{c} p \\ 2k \end{array}\right) A^{p-(2k+1)} \left(-19B^2\right)^k.$$

So $A = \pm 1$. We get $y = A^2 + 19B^2 = 1 + 19B^2$. Since $2 \nmid y$, we have $2 \mid B$. Therefore, $x \geq 1$, from $19x^2 + 1 = y^p$ we have $y > 1$. Hence from $y = 1 + 19B^2$, we deduce $B \neq 0$. On the other hand

$$1 = A_{k=0}^{\frac{p-1}{2}} \left(\begin{array}{c} p \\ 2k \end{array}\right) A^{p-(2k+1)} \left(-19B^2\right)^k \equiv A^p \,(mod 19).$$

Since $A = 1$, we obtain

$$0 =_{k=1}^{\frac{p-1}{2}} \left(\begin{array}{c} p \\ 2k \end{array}\right) \left(-19B^2\right)^k.$$

Let $V_2(.)$ be the standart 2-adic valuation. For $k \geq 2$, let $k = 2^s t,\ 2 \nmid t$. Then when $s = 0$ we have $k = t \geq 2$ and $V_2(k) = s = 0$. So $2(k-1) = 2(t-1) \geq 2 > 0 = V_2(k)$ and when $s > 0, 2(k-1) = 2(2^s t - 1) \geq 2(2^s - 1) \geq 2s > s = V_2(k)$. Since $2 \mid B$ and $B \neq 0$ for $1 < k \leq \frac{p-1}{2}$ we have

$$V_2\left(\binom{p}{2k}(-19B^2)^k\right) = V_2\left(\frac{p(p-1)}{2k(2k-1)}\binom{p-2}{2k-2}(-19B^2)^k\right)$$
$$\geq V_2\left(\binom{p}{2}(-19B^2)\right) + 2(k-1) - V_2(k)$$
$$> V_2\left(\binom{p}{2}(-19B^2)\right)$$

But from $0 = \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{2k}(-19B^2)^k$, we see that there are at least two terms with the smallest 2-adic valuation. This is a contradiction. ■

2 The equation $x^2 + 19^{2k} = y^n$

Now, we can give our main theorem for m is an even;

Theorem 2 *The equation $x^2 + 19^{2k} = y^n$, with $n > 2$ and $k > 0$ has no positive integer solution.*

Proof. For proof, we must consider three different cases. ■

The first case is, for $n \geq 4$ is an even.

Lemma 3 *If $3 \mid k$ or $q \equiv \pm 3 \mod 8$ then the Diophantine equation $x^2 + q^{2k} = y^n$ where $n \geq 4$ is an even, q is an odd prime and $(q, x) = 1$, has no solution* [1].

From this Lemma we can say when $n \geq 4$ is an even, the equation $x^2 + 19^{2k} = y^n$ has no positive integer solution.

The second case is, $n = 3$.

Claim 4 *The equation $x^2 + 19^{2k} = y^3,\ k > 0$ has no positive integer solution.*

Firstly, we assume that $(x, 19) = 1$. Since x and y are coprime and $(x, 19) = 1$, then we get x is an even. By factorization, the equation becomes

$$\left(x + 19^k i\right)\left(x - 19^k i\right) = y^3.$$

$\left(x + 19^k i\right)$ and $\left(x - 19^k i\right)$ are coprime in $\mathbb{Z}[i]$. Because of the units of $\mathbb{Z}[i]$ are only $\pm 1,\ \pm i$, we get

$$2i19^k = (u + iv)^3 - (u - iv)^3$$

or

$$19^k = v\left(3u^2 - v^2\right).$$

Note that u and v are coprime, otherwise any prime factor of u and v will also divide both x and y. Therefore, $v = \pm 1$ or $v = \pm 19^k$ which leads to:

$$19^k = \pm 1\left(3u^2 - 1\right) \quad \text{or} \quad 19^k = \pm 19^k\left(3u^2 - 19^{2k}\right)$$

$$\pm 19^k = 3u^2 - 1 \qquad\qquad 1 = \pm 1\left(3u^2 - 19^{2k}\right)$$

$$3u^2 = 1 \pm 19^k \qquad\qquad 3u^2 = \pm 1 + 19^{2k}$$

Now we consider the first equation, if we consider the sign is negative, then the right hand side gives negative value, so this is impossible. If we think the sign is $+$ and k is an even, then the right hand side is congruent to 2 *modulo* 3 while the left hand side is divisible by 3 which is a contradiction. Finally if we consider the sign is $+$ and k is odd, then we reach similar contradiction. So both of the two cases have no solution.

The second equation the sign must be -1. Thus $\left(19^k\right)^2 - 3u^2 = 1$ i.e. $X^2 - 3Y^2 = 1$ has a fundamental solution $(X_1, Y_1) = (2, 1)$. Furthermore $X_2 = 7,\ X_3 = 26...$. (X_m) with the recurrence sequence $X_m = 4X_{n-2} - X_{n-1}$ is a Lucas Sequence of second type. By Primitive Divisor Theorem [4] for $m > 12$ and checking X_m for all ≤ 12, we get X_m cannot be power of 19.

Now we assume $19 \mid x$. Let $x = 19^u.X,\ \ y = 19^v.Y$ where $u > 0,\ \ v > 0$ and $(19, X) = (19, Y) = 1$. Then equation becomes

$$\left(19^u X\right)^2 + 19^{2k} = 19^{3v} Y^3 \tag{2.1}$$

and we have three possibilities.

1) $2u = min\,(2u, 3v, 2k)$. By cancelling 19^{2u} in (2.1) we get

$$X^2 + 19^{2(k-u)} = 19^{3v-2u} Y^3.$$

If $k - u = 0$ and $3v - 2u = 0,$ we get the famous equation of Lebesque which has no solution $[10]$.

If $k - u > 0$ and $3v - 2u = 0,$ then considering *modulo* 19, we get the equation in Claim 4 with the first assume. We know with this assume the equation $x^2 + 19^{2k} = y^3$ has no solution.

Finally if $k - u = 0$, we get

$$X^2 + 1 = 19^{3v-2k} Y^3 \tag{2.2}$$

If $3 \mid k$ then we can write the equation (2.2) as

$$X^2 + 1 = (19^{v - \frac{2k}{3}} Y)^p$$

which has no solution.

2) $2k = min\,(2u, 3v, 2k)$. By cancelling 19^{2k} in (2.1) we get

$$\left(19^{u-k} X\right)^2 + 1 = 19^{3v-2k} Y^3.$$

Considering this equation *modulo* 19 we get either $3v - 2k = 0$ i.e. $\left(19^{u-k} X\right)^2 + 1 = Y^3$ which has no solution from *Lemma* 1 or $3v - 2k > 0$ namely we get the same equation in (2.2).

3) $3v = min\,(2u, 3v, 2k)$
This case do not give us any solution. This completes the proof.

The last case is, $n = p \geq 5$ where p is a prime.

Claim 5 *The equation $x^2 + 19^{2k} = y^p,\ k > 0$ has no positive integer solution.*

Fistly we assume that $(x, 19) = 1$. A. Berczes and I. Pink [3] proved that there are no solution to the equation $x^2 + p^{2k} = y^n$, with $p \in \{19, 41, 59, 61, 79\}$, $n \geq 5$ and $k \geq 3,\ (x, y) = 1$.

Now we assume $19 \mid x$. Let $x = 19^u.X,\ y = 19^v.Y$ where $u > 0,\ v > 0$ and $(19, X) = (19, Y) = 1$. Then, equation becomes

$$(19^u X)^2 + 19^{2k} = 19^{pv} Y^p \tag{2.3}$$

and we have three possibilities.

1) $2u = min\,(2u, pv, 2k)$. By cancelling 19^{2u} in (2.3) we get

$$X^2 + 19^{2(k-u)} = 19^{pv-2u} Y^p.$$

If $k - u = 0$ and $pv - 2u = 0$, we get the famous equation of Lebesque which has no solution $[10]$.

If $k - u > 0$ and $pv - 2u = 0$, then considering $modulo$ 19, we get Berczes and Pink's equation $[3]$. We know this equation has no solution.

Finally, if $k - u = 0$, we get

$$X^2 + 1 = 19^{pv-2k} Y^p \tag{2.4}$$

If $p \mid k$ then we can write the equation (2.4) as

$$X^2 + 1 = (19^{v - \frac{2k}{p}} Y)^p$$

which has no solution.

2) $2k = min\,(2u, pv, 2k)$. By cancelling 19^{2k} in (2.3) we get

$$\left(19^{u-k} X\right)^2 + 1 = 19^{pv-2k} Y^p.$$

Considering this equation $modulo$ 19 we get either $pv - 2k = 0$ i.e. $\left(19^{u-k} X\right)^2 + 1 = Y^p$ which has no solution from $Lemma$ 1 or $pv - 2u > 0$ namely we get the same equation in (2.4).

3) $pv = min\,(2u, pv, 2k)$
This case do not give us any solutions.
All of the cases complete the proof of the Theorem 2.

3 The equation $x^2 + 19^{2k+1} = y^n$

For the equation $x^2 + 19^{2k+1} = y^n$ the case $k = 0$, $n \geq 3$ has been studied by Cohn [6] and the case $k \geq 0$, $n \geq 5$ has been studied by Arif and Muriefah [2]. Therefore, we consider the case $k > 0$ and $n = 3, 4$.

Now, we can give our main theorem for m is an odd.

Theorem 6 *The equation* $x^2 + 19^{2k+1} = y^n$*, where* $n = 3, 4$ *and* $k > 0$ *has no positive integer solution.*

Proof. For proof we consider the cases $n = 3$ and $n = 4$ separately. ■

The case $n = 3$**,** $k > 0$

For this case, we first assume $(19, x) = 1$. There is no loss generality in considering only $n = 3$ is an odd prime. Since the class number of the field $\mathbb{Q}\left(\sqrt{-19}\right)$ is not multiply by $n = 3$, we have

$$x + 19^k\sqrt{-19} = \left(\frac{a + b\sqrt{-19}}{2}\right)^3$$

where $y = \dfrac{a^2 + 19b^2}{4}$ for some rational integers a and b. Equating imaginary parts, we get

$$\begin{aligned} 19^k.2^3 &= b.\left[\binom{3}{1}a^2 + \binom{3}{3}\left(-19b^2\right)\right] \\ 19^k.8 &= b.\left(3a^2 - 19b^2\right). \end{aligned}$$

For this equation we have three possibilities. If we take $b = \pm 1$, then we obtain

$$\pm 19^k.8 = 3a^2 - 19.$$

There is no solution of this equation.

If we take $b = \pm 19^k$ then we get

$$\pm 8 = 3a^2 - 19^{2k+1} \tag{3.1}$$

where $k > 0$. There is no solution of this equation, too.

If we take $b = \pm 19^\lambda$ $(0 < \lambda < k)$ then we get

$$\pm 19^{k-\lambda}.8 = 3a^2 - 19^{2\lambda+1}.$$

If $k - \lambda > 0$, this is not possible *modulo* 19. If $k - \lambda = 0$ that is $k = \lambda$, then we get the equation (3.1) again and we know this equation doesn't have a solution, where $k > 0$. So there is no solution of this equation.

Secondly we assume $19 \mid x$. Let $x = 19^s.X$, $y = 19^t.Y$ where $s > 0$, $t > 0$ and $(19, X) = (19, Y) = 1$. Then, equation becomes

$$\left(19^s X\right)^2 + 19^{2k+1} = 19^{3t} Y^3 \tag{3.2}$$

and we have three possibilities.

1) $2s = min\,(2s, 3t, 2k+1)$. By cancelling 19^{2s} in (3.2) we get

$$X^2 + 19^{2(k-s)+1} = 19^{3t-2s}Y^3$$

and considering this equation $modulo$ 19.

$$X^2 + 19^{2(k-s)+1} = Y^3$$

We deduce that $3t - 2s = 0$, i.e. $3t = 2s$ then $3 \mid s$, $s = 3M$. For $k - s > 0$, this equation has no solution.

2) $2k+1 = min\,(2s, 3t, 2k+1)$. Then we get

$$19^{2s-2k-1}X^2 + 1 = 19^{3t-2k-1}Y^3$$

and considering this equation $modulo$ 19, we get $3t - 2k - 1 = 0$, so that

$$19\ \left(19^{s-k-1}X\right)^2 + 1 = Y^3$$

we obtain

$$19Z^2 + 1 = Y^3$$

from $Lemma$ 1 we can say this equation has no positive integer solution.

3) $3t = min\,(2s, 3t, 2k+1)$. Then, we get

$$19^{2s-3t}X^2 + 19^{2k+1-3t} = Y^3$$

this is impossible $modulo$ 19 only if $2s - 3t = 0$ or $2k + 1 - 3t = 0$ and both of these cases have already been discussed. This includes the proof of the theorem's first case.

The case $n = 4$**,** $k > 0$

When $n = 4$ we have $x^2 + 19^{2k+1} = y^4$. If y is an even, then $y^4 \equiv 0 \mod 8$. If we reduce the equation for $modulo$ 8, then we get $x^2 + 3 \equiv 0 \mod 8$ namely $x^2 \equiv 5 \mod 8$. From Legendre symbol $\left(\frac{5}{8}\right) = -1$, this equation is not solvable. So y is an odd and x is an even due to $(x, y) = 1$. We have $19^{2k+1} = (y^4 - x^2) = (y^2 - x)(y^2 + x)$ .Then

$$y^2 - x = 1$$

$$y^2 + x = 19^{2k+1}$$

so

$$2y^2 = 19^{2k+1} + 1.$$

Then, $2y^2 \equiv 4 \mod 8$ i.e. $y^2 \equiv 2 \mod 4$, which is impossible.

This completes proof of the Theorem 6.

References

[1] Abu Muriefah, F. S., S.A. Arif, The Diophantine equation $x^2 + q^{2k} = y^n$, *The Arabian J. for Sci. and Engineering*, Vol. 26, 2001, No. 1A, 53–62.

[2] Arif, S. A., F.S. Abu Muriefah, On the Diophantine equation $x^2 + q^{2k+1} = y^n$, *Journal of Number Theory*, Vol. 95, 2002, No. 1, 95–100.

[3] Berczes, A., I. Pink, On the Diophantine equation $x^2 + p^{2k} = y^n$, *Arch. der Math.*, Vol. 91, 2008, 505–517.

[4] Bilu, Y. F., G. Hanrot and P.M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, *J. Reine Angew. Math.*, Vol. 539, 2001, 75–122.

[5] Cangül, İ. N., G. Soydan and Y. Şimşek, A p-adic look at the Diophantine equation $x^2 + 11^{2k} = y^n$, *Numerical Analysis and Applied Mathematics, International Conference* Vol. 1, 2009, 275–277.

[6] Cohn, J. H. E., The Diophantine equation $x^2 + 19 = y^n$, *Acta Arith.*, Vol. LXI, 1992, No. 2, 193–197.

[7] Cohn, J. H. E., On the equation $x^4 - Dy^2 = 1$ II. *Acta Arith.*, Vol. LXV, 1993, No. 4, 367–381.

[8] Cohn, J. H. E., The Diophantine equation $x^2 + C = y^n$, *Acta Arith.*, Vol. 65, 1995, No 4., 367–381.

[9] Demirpolat, E., S. (İnağ) Çenberci, H. Şenay, On the Diophantine equation $x^2 + 11^{2k+1} = y^n$, *International Mathematical Forum*, Vol. 6, 2009, No. 4, 277–280.

[10] Lebesque, V. A. Sur l'impossibilite en nombres entries de l'equation $x^m = y^2+1$, *Nouvelles Annales des Mathematiques*, Vol. 9, 1850, No. 1, 178–181.

[11] Levesque, C. On a few Diophantine equations, in particular, Fermat's Last Theorem, *IJMMS* Vol. 71, 2003, 4473–4500.

[12] Mordell, L. J. *Diophantine Equations*, Academic Press, London, 1969.

[13] Tao, L. On the Diophantine equation $x^2 + 5^m = y^n$, *Ramanujan Journal*, Vol. 19, 2009, 325–338.

[14] Zhu, H., M. Le, On some generalized Lebesque–Nagell equations, *Journal of Number Theory*, Vol. 131, 2011, 458–469.