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Notes on efficient computation of Ramanujan cubic equations

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Abstract: This paper considers properties of a theorem of Ramanujan to develop properties and algorithms related to cubic equations. The Ramanujan cubics are related to the Cardano cubics and Padovan recurrence relations. These generate cubic identities related to heptagonal triangles and third order recurrence relations, as well as an algorithm for finding the real root of the relevant Ramanujan cubic equation. The algorithm is applied to, and analyzed for, some of the earlier examples in the paper.

Keywords: Ramanujan-type identity, Cubic equation, Trigonometric functions.
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1 Introduction

The purpose of this paper is to extend a theorem of Ramanujan to develop associated cubic equations with an algorithm for their solution. In particular, the motivation is to compute Ramanujan-type identities. The Ramanujan's theorem is enunciated in Section 1. By utilizing Ramanujan's theorem and Cardano's formula a number of Ramanujan-type identities are listed. The main results are then established in three theorems in Section 2. Section 3 describes an efficient and effective algorithm for finding the real root of an associated Ramanujan cubic, while Section 4 illustrates the algorithm applied to examples arising in the preceding sections. Section 5 analyzes the two algorithms which are compared in the paper. Section 6 provides further propositions which incidentally contain particular second and third order recurrence relations. It is interesting to have an explicit expression of t in terms of the coefficients a and b based on which an algorithm can be devised to determine t. This algorithm is more efficient than the one given by Wang, which relies on the three roots of the cubic equation $f(x)$.

To put these in a broader number theoretic context, these Ramanujan cubics are in turn related to the Cardano cubics and Vieta's formulas [10] and Padovan recurrence relations [6] which have applications with the so-called plastic numbers [15]. The identities build on trigonometric functions and relate to heptagonal triangles; these relate to particular cases of third order recurrence relations [14].

Next, consider the following theorem of Ramanujan [13]:

Theorem 1.1. *Let $\alpha,\ \beta$, and γ denote the distinct roots of a cubic equation*

$$x^3 - ax^2 + bx - 1 = 0. \tag{1}$$

If $\alpha,\ \beta$, and γ are real and distinct and the cubic roots of these numbers below are real, then

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a + 6 + 3t} \quad \textit{and} \tag{2}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{b + 6 + 3t}, \tag{3}$$

where t is the only real root of the associated Ramanujan equation

$$t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0 \tag{4}$$

with $t \neq \alpha,\ \beta,\ \gamma$.

A simple and elegant proof of Theorem 1.1 was given by Berndt and Bhargava [2]. Applying this theorem, the following Ramanujan's identities can be obtained [13]:

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}, \tag{5}$$

$$\sqrt[3]{\cos\frac{2\pi}{9}} + \sqrt[3]{\cos\frac{4\pi}{9}} + \sqrt[3]{\cos\frac{8\pi}{9}} = -\sqrt[3]{\frac{6 - 3\sqrt[3]{9}}{2}}, \tag{6}$$

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}, \tag{7}$$

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}}=-\sqrt[3]{6-6\sqrt[3]{9}} \tag{8}$$

Wang [22], inspired by Ramanujan's identities involving cosine functions, investigated similar identities involving sine and tangent functions. By using Ramanujan's theorem and Cardano's formula, he provided the following identities:

$$\begin{aligned}&\sqrt[3]{\sin\frac{2\pi}{7}}+\sqrt[3]{\sin\frac{4\pi}{7}}+\sqrt[3]{\sin\frac{8\pi}{7}}\\&\quad=\left(-\sqrt[18]{\frac{7}{64}}\right)\left(\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right),\end{aligned} \tag{9}$$

$$\begin{aligned}&\frac{1}{\sqrt[3]{\sin\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\sin\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\sin\frac{8\pi}{7}}}\\&\quad=\left(-\sqrt[18]{\frac{64}{7}}\right)\left(\sqrt[3]{6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right),\end{aligned} \tag{10}$$

$$\sqrt[3]{\sin\frac{\pi}{9}}+\sqrt[3]{\sin\frac{2\pi}{9}}+\sqrt[3]{\sin\frac{14\pi}{9}}=-\frac{\sqrt[18]{3}}{2}\sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}, \tag{11}$$

$$\frac{1}{\sqrt[3]{\sin\frac{\pi}{9}}}+\frac{1}{\sqrt[3]{\sin\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\sin\frac{14\pi}{9}}}=-\frac{2}{\sqrt[18]{3}}\sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}, \tag{12}$$

$$\begin{aligned}&\sqrt[3]{\tan\frac{2\pi}{7}}+\sqrt[3]{\tan\frac{4\pi}{7}}+\sqrt[3]{\tan\frac{8\pi}{7}}\\&\quad=\left(\sqrt[18]{7}\right)\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)},\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{1}{\sqrt[3]{\tan\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\tan\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\tan\frac{8\pi}{7}}}\\&\quad=\left(-\frac{1}{\sqrt[18]{7}}\right)\sqrt[3]{-\sqrt[3]{49}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)},\end{aligned} \tag{14}$$

$$\begin{aligned}&\sqrt[3]{\tan\frac{\pi}{9}}+\sqrt[3]{\tan\frac{4\pi}{9}}+\sqrt[3]{\tan\frac{7\pi}{9}}\\&\quad=\left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3}+6+3\left(\sqrt[3]{21-3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}-\sqrt[3]{3+3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}\right)}\right),\end{aligned} \tag{15}$$

$$\frac{1}{\sqrt[3]{\tan\frac{\pi}{9}}}+\frac{1}{\sqrt[3]{\tan\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\tan\frac{7\pi}{9}}}$$
$$=\left(-\frac{1}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{21-3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}-\sqrt[3]{3+3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}\right)}\right). \tag{16}$$

Remark 1.1. *The authors note that in* (15) *and* (16) *there was a typographical error in [22] which is presented in the correct way in this paper.*

To verify the roots are of the cubic equation defined in the examples, one may use Vieta's formula [8]. Wang's proof for these identities involves large amount of calculations. In particular, his computation is based on his Theorem 1.7, which relies on computing

$$\left(\left(\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}\right)-\left(\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}\right)\right)^2,$$

where α, β, and γ are the roots of $x^3-ax^2+bx-1=0$.

This paper uses the work of Liao, Saul, and Shiue [11], and Chen [5] to determine the root t from the associated Ramanujan equation (4), with t being the only one real root; it is proved in Theorem 2.3. Liao et al. and Chen's work are based on Sylvester [17]. The next theorem is a result from [11].

Theorem 1.2. *Given a polynomial equation of degree three $x^3+px+q=0$ $(p,q\neq 0)$, with real coefficients. Let $p=-3rs$ and $q=rs(r+s)$. Then the three solutions to this equation are*

$$x=-r^{\frac{1}{3}}s^{\frac{1}{3}}\left(r^{\frac{1}{3}}+s^{\frac{1}{3}}\right),\ -r^{\frac{1}{3}}s^{\frac{1}{3}}\left(\omega r^{\frac{1}{3}}+\omega^2 s^{\frac{1}{3}}\right),\ -r^{\frac{1}{3}}s^{\frac{1}{3}}\left(\omega^2 r^{\frac{1}{3}}+\omega s^{\frac{1}{3}}\right),$$

where $\omega=\dfrac{-1+\sqrt{3}i}{2}$.

Moreover, if $r,s\in\mathbb{R}$ and $r\neq s$, then the equation has one real root and a pair of complex conjugate roots.

We will use this theorem, combining it with Theorem 1.1, to provide an efficient method to Wang's results and other Ramanujan-type identities.

We present our main results in the next section. Examples are given in Section 4. An analysis of algorithms based on Wang's method and our method is shown in Section 3.

2 Main results

Theorem 2.1. *Let $f(x)=x^3-ax^2+bx-1=0$, $a,b\in\mathbb{R}$, a,b not both 0. Let $f(x)$ have three distinct real roots α, β, and γ. Let $t^3+pt+q=0$ be the associated Ramanujan equation, where $p=-3(a+b+3)$ and $q=-(ab+6(a+b)+9)$. Then*

$$t=\sqrt[3]{\frac{ab+6(a+b)+9+\Delta}{2}}+\sqrt[3]{\frac{ab+6(a+b)+9-\Delta}{2}}, \tag{17}$$

where the discriminant is $\Delta^2 = (ab)^2 - 4(a^3+b^3) + 18ab - 27$. *Moreover,*

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a+6+3t},$$
$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{b+6+3t}.$$

Proof. Using Liao et al.'s method [11], we have

$$-3rs = -3(a+b+3) \implies rs = a+b+3, \tag{18}$$
$$rs(r+s) = -(ab+6(a+b)+9) \implies r+s = -\frac{(ab+6(a+b)+9)}{rs}. \tag{19}$$

To compute r and s, we compute $r-s$. Then

$$\begin{aligned}(r-s)^2 = (r+s)^2 - 4rs &= \frac{1}{(rs)^2}\left((ab+6(a+b)+9)^2 - 4(a+b+3)^3\right)\\ &= \frac{1}{(rs)^2}\Big[(ab)^2 + 12a^2b + 36a^2 + 12ab^2 + 90ab + 108a + 36b^2 + 108b + 81\\ &\quad - 4\left(a^3 + 3a^2b + 9a + 3ab^2 + 18ab + 27a + b^3 + 9b^2 + 27b + 27\right)\Big]\\ &= \frac{1}{(rs)^2}\left((ab)^2 - 4\left(a^3+b^3\right) + 18ab - 27\right)\\ &\implies (r-s)^2(rs)^2 = (ab)^2 - 4\left(a^3+b^3\right) + 18ab - 27.\end{aligned}$$

Let $\Delta^2 = (ab)^2 - 4(a^3+b^3) + 18ab - 27 = (r-s)^2(rs)^2$. Note that the discriminant of $t^3+pt+q=0$ is always a perfect square. Then

$$r - s = \frac{\Delta}{rs}.$$

Hence,

$$2r = -\frac{ab+6(a+b)+9}{rs} + \frac{\Delta}{rs} \implies r^2s = \frac{-(ab+6(a+b)+9)+\Delta}{2},$$
$$2s = -\frac{ab+6(a+b)+9}{rs} - \frac{\Delta}{rs} \implies rs^2 = \frac{-(ab+6(a+b)+9)-\Delta}{2}.$$

Thus,

$$\begin{aligned}t &= -(rs)^{\frac{1}{3}}\left(r^{\frac{1}{3}} + s^{\frac{1}{3}}\right) = -\left(r^{\frac{2}{3}}s^{\frac{1}{3}} + r^{\frac{1}{3}}s^{\frac{2}{3}}\right)\\ &= \sqrt[3]{\frac{(ab+6(a+b)+9)+\Delta}{2}} + \sqrt[3]{\frac{(ab+6(a+b)+9)-\Delta}{2}}.\end{aligned}$$

From Theorem 2.1, we have the second result. □

Corollary 2.1. (Shanks Polynomial [7]) *Let* $f(x) = x^3 - ax^2 - (a+3)x - 1 = 0$. *The associated Ramanujan equation is* $t^3 + q = 0$, *where* $q = (a(a+3)+9)$. *Then* $t = -\sqrt[3]{a^2+3a+9}$. *The roots of* $f(x)$ *are*

$$\frac{1}{3}\left(a + 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right) + k\pi\right)\right)\right) \quad \textit{if} \quad a \geq -\frac{3}{2},$$
$$\frac{1}{3}\left(a - 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right) + k\pi\right)\right)\right) \quad \textit{if} \quad a \leq -\frac{3}{2},$$

where $k = 0, 2, 4$. *Then by Theorem 2.1, we have for* $a \geq -\frac{3}{2}$, $k = 0, 2, 4$,

$$\sum \sqrt[3]{\frac{1}{3}\left(a+2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)} = \sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}$$

$$\implies \sum \sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)} = \sqrt[3]{\frac{3}{a+2\sqrt{a^2+3a+9}}}\sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}.$$

For $a \le -\frac{3}{2}$, $k = 0, 2, 4$, *we have*

$$\sum \sqrt[3]{\frac{1}{3}\left(a-2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)} = \sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}$$

$$\implies \sum \sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)} = \frac{\sqrt[3]{3}}{\sqrt[3]{a-2\sqrt{a^2+3a+9}}}\sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}.$$

On the other hand, for $a \ge -\frac{3}{2}$, $k = 0, 2, 4$,

$$\sum \frac{1}{\sqrt[3]{\frac{1}{3}\left(a+2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)}} = \sqrt[3]{-(a+3)+6-3\sqrt[3]{a^2+3a+9}}$$

$$\implies \sum \frac{1}{\sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)}} = \sqrt[3]{\frac{a+2\sqrt{a^2+3a+9}}{3}}\sqrt[3]{-a+3-3\sqrt[3]{a^2+3a+9}}.$$

For $a \le -\frac{3}{2}$, $k = 0, 2, 4$, *we have*

$$\sum \frac{1}{\sqrt[3]{\frac{1}{3}\left(a-2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)}} = \sqrt[3]{-(a+3)+6-3\sqrt[3]{a^2+3a+9}}$$

$$\implies \sum \frac{1}{\sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)}} = \sqrt[3]{\frac{a-2\sqrt{a^2+3a+9}}{3}}\sqrt[3]{-a+3-3\sqrt[3]{a^2+3a+9}}.$$

Corollary 2.2. *From Theorem 2.1, if* $a+b+3=0$*, then* $t = \sqrt[3]{ab-9}$.

The condition $a+b+3=0$ was researched extensively by several authors [2,7,16]. Examples with this condition will be given in Section 4.

Theorem 2.2. *Let* $f(x) = x^3 - ax^2 + bx - 1 = 0$ *with* $a+b+3=0$*. Then all three roots are distinct.*

Proof. From Theorem 2.1, the discriminant of $f(x) = 0$ is:

$$\Delta^2 = (ab)^2 - 4(a^3+b^3) + 18ab - 27.$$

Then:

$$\begin{aligned} D &= a^2b^2 - 4b^3 - 4a^3 + 18ab - 27 = a^2b^2 - 4(a^3+b^3) + 18ab - 27 \\ &= a^2b^2 - 4(a+b)(a^2-ab+b^2) + 18ab - 27 = a^2b^2 + 12(a^2-ab+b^2) + 18ab - 27 \\ &= a^2b^2 + 12(a^2+b^2) + 6ab - 27 = a^2b^2 + 12[(a+b)^2 - 2ab] + 6ab - 27 \\ &= a^2b^2 + 12(9-2ab) + 6ab - 27 = a^2b^2 - 18ab + 81 = (ab-9)^2 \ge 0. \end{aligned}$$

Therefore $f(x) = 0$ with $a+b+3=0$ has three distinct roots if $ab \ne 9$. If $ab = 9$, then combined with $a+b+3=0$, $b^2+3b+9=0 \implies b$ is not real. $\square$

Corollary 2.3. *Let* $f(x) = x^3 - ax^2 + bx - 1$ *with* $a + b + 3 = 0$*. The roots of* $f(x)$ *are the negatives of* $g(x) = x^3 + ax^2 + bx + 1$ *with* $a + b + 3 = 0$*. Then* $g(x) = 0 = g\left(\frac{1}{1-x}\right), x \neq 1$*.*

Proof.

$$\begin{aligned}
g\left(\frac{1}{1-x}\right) &= \left(\frac{1}{1-x}\right)^3 + a\left(\frac{1}{1-x}\right)^2 + b\left(\frac{1}{1-x}\right) + 1 \\
&= \frac{1}{(1-x)^3}[1 + a(1-x) + b(1-x)^2 + (1-x)^3] \\
&= \frac{1}{(1-x)^3}[1 + a - ax + b - 2bx + bx^2 + (1 - 3x + 3x^2 - x^3)] \\
&= \frac{1}{(1-x)^3}[-x^3 + (3+b)x^2 - (a + 2b + 3)x + (1 + a + b + 1)] \\
&= \frac{1}{(1-x)^3}[-x^3 - ax^2 - bx - 1] = 0 \\
&\implies x^3 + ax^2 + bx + 1 = 0 \\
&\implies g\left(\frac{1}{1-x}\right) = 0 = g(x).
\end{aligned}$$

□

Theorem 2.3. *Let* $t^3 + pt + q = 0$ *be a cubic equation, where* $p = -3(a + b + 3)$ *and* $q = -(ab + 6(a + b) + 9)$*,* $a, b \in \mathbb{R}$*,* a, b *not both* 0*. Then only one root is real.*

Proof. Using Theorem 1.2 and the proof of Theorem 2.1, we see that $r \neq s$. Thus, $f(x)$ has only one real root. □

3 Algorithms

In this section, two algorithms are given. The first algorithm is based on Wang's approach to finding the root of the associated Ramanujan equation $t^3 + pt + q = 0$. The second algorithm is our approach. Again, denote $f(x) = x^3 - ax^2 + bx - 1$.

Algorithm 1 Algorithm for Wang's approach

Input: a, b from $f(x) = 0$

Output: $t \in \mathbb{R}$, root of $t^3 + pt + q = 0$.

Find the roots of a cubic equation $f(x) = 0$: α, β, γ
$A_1 \leftarrow \alpha/\beta$
$A_2 \leftarrow \beta/\gamma$
$A_3 \leftarrow \gamma/\alpha$
$B \leftarrow A_1 + A_2 + A_3$
$C \leftarrow 1/A_1 + 1/A_2 + 1/A_3$
$D \leftarrow (B - C)^2$
$t \leftarrow \sqrt{D}$
return t

Given a cubic equation, Algorithm 1 requires finding the roots of $f(x) = 0$ first. In the next three steps, we compute the ratios $\frac{\alpha}{\beta}$, $\frac{\beta}{\gamma}$, $\frac{\gamma}{\alpha}$, and their reciprocals. We sum these and compute the square of the difference. Taking the square root of this result will give the real root t.

Algorithm 2 Algorithm for our approach

Input: a, b from $f(x) = 0$

Output: $t \in \mathbb{R}$, root of $t^3 + pt + q = 0$

$A \leftarrow ab + 6(a+b) + 9$

$B \leftarrow (ab)^2 - 4(a^3 + b^3) + 18ab - 27$

$C \leftarrow \sqrt{B}$

$D \leftarrow (A + C)/2$

$E \leftarrow (A - C)/2$

$t \leftarrow \sqrt[3]{D} + \sqrt[3]{E}$

return t

In our approach, we need only the coefficients a and b to compute the only real root t. Note that in the first step, $ab + 6(a+b) + 9$ is the negative of the constant term q of the associated Ramanujan equation. To obtain Ramanujan-type identities, we also need to find the roots of $f(x) = 0$ by using Vieta's formula. We will demonstrate the application of our algorithm in the next section. The detailed analysis of both algorithms will be given in Section 5.

4 Examples

In this section, several examples from [22] are presented. Theorem 2.1 is used to provide an alternate method to [22]. In each example, the roots of the cubic equation can be verified using Vieta's formula.

Our computation here does not require the use of the roots α, β, and γ, which simplifies the computation process. More generally, t can be computed more efficiently in our new theorem.

We will use Algorithm 2 in the following examples.

4.1 Equation $x^3 - ax^2 + bx - 1 = 0, a + b + 3 \neq 0$

Example 4.1. ([22]). *Let $f(x) = x^3 - \sqrt[3]{9}x - 1 = 0$. Denote $a = 0$ and $b = -\sqrt[3]{9}$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = 3\sqrt[3]{9} - 9$ and $q = 6\sqrt[3]{9} - 9$. Following the steps in Algorithm 2, then*

$$ab + 6(a+b) + 9 = A = -6\sqrt[3]{9} + 9.$$

Next,

$$\begin{aligned} \Delta^2 = B &= (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = -4(-9) - 27 = 9 \\ \implies \Delta = C &= 3, \end{aligned}$$

and

$$\begin{aligned} D &= \frac{1}{2}(A + C) = \frac{1}{2}\left(-6\sqrt[3]{9} + 9 + 3\right) = -3\sqrt[3]{9} + 6, \\ E &= \frac{1}{2}(A - C) = \frac{1}{2}\left(-6\sqrt[3]{9} + 9 - 3\right) = -3\sqrt[3]{9} + 3. \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}.$$

The roots of the cubic equation $x^3 - \sqrt[3]{9}x - 1 = 0$*, proved in [22], are* $\alpha = -\frac{2}{\sqrt[6]{3}} \sin \frac{\pi}{9}$*,* $\beta = -\frac{2}{\sqrt[6]{3}} \sin \frac{2\pi}{9}$*, and* $\gamma = -\frac{2}{\sqrt[6]{3}} \sin \frac{14\pi}{9}$*. Then by Theorem 1.1,*

$$\sqrt[3]{\sin \frac{\pi}{9}} + \sqrt[3]{\sin \frac{2\pi}{9}} + \sqrt[3]{\sin \frac{14\pi}{9}} = -\frac{\sqrt[18]{3}}{2} \sqrt[3]{6 + 3\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)},$$

$$\frac{1}{\sqrt[3]{\sin \frac{\pi}{9}}} + \frac{1}{\sqrt[3]{\sin \frac{2\pi}{9}}} + \frac{1}{\sqrt[3]{\sin \frac{14\pi}{9}}} = -\frac{2}{\sqrt[18]{3}} \sqrt[3]{-\sqrt[3]{9} + 6 + 3\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)}.$$

Example 4.2. ([22]) *Let* $f(x) = x^3 + 3\sqrt[3]{3}x^2 - \sqrt[3]{9}x - 1 = 0$*. Denote* $a = -3\sqrt[3]{3}$ *and* $b = -\sqrt[3]{9}$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 3\left(3\sqrt[3]{3} + \sqrt[3]{9} - 3\right)$ *and* $q = -\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)\right)$*. Following the steps in Algorithm 2, then*

$$\begin{aligned} ab + 6(a + b) + 9 = A &= -3\sqrt[3]{3}\left(-\sqrt[3]{9}\right) + 6\left(-3\sqrt[3]{3} - \sqrt[3]{9}\right) + 9 \\ &= 18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right). \end{aligned}$$

Next,

$$\begin{aligned} &\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 81 + 4(81 + 9) + 162 - 27 = 576 \\ \implies &\Delta = C = 24, \end{aligned}$$

and

$$\begin{aligned} D &= \frac{1}{2}(A + C) = \frac{1}{2}\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right) + 24\right) = 21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right), \\ E &= \frac{1}{2}(A - C) = \frac{1}{2}\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right) - 24\right) = -3 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right). \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} + \sqrt[3]{-3 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}.$$

The roots of $x^3 + 3\sqrt[3]{3}x^2 - \sqrt[3]{9}x - 1 = 0$*, proved in [22], are* $\alpha = -\frac{1}{\sqrt[6]{3}} \tan \frac{\pi}{9}$*,* $\beta = -\frac{1}{\sqrt[6]{3}} \tan \frac{4\pi}{9}$*, and* $\gamma = -\frac{1}{\sqrt[6]{3}} \tan \frac{7\pi}{9}$*. Then, by Theorem 1.1,*

$$\begin{aligned} &\sqrt[3]{\tan \frac{\pi}{9}} + \sqrt[3]{\tan \frac{4\pi}{9}} + \sqrt[3]{\tan \frac{7\pi}{9}} \\ &\quad = \left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3} + 6 + 3\left(\sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} - \sqrt[3]{3 + 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}\right)}\right), \end{aligned}$$

$$\begin{aligned} &\frac{1}{\sqrt[3]{\tan \frac{\pi}{9}}} + \frac{1}{\sqrt[3]{\tan \frac{4\pi}{9}}} + \frac{1}{\sqrt[3]{\tan \frac{7\pi}{9}}} \\ &\quad = \left(-\frac{1}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9} + 6 + 3\left(\sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} - \sqrt[3]{3 + 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}\right)}\right). \end{aligned}$$

Example 4.3. ([22]) *Let* $f(x) = x^3 + \sqrt[3]{7}x^2 - 1 = 0$. *Denote* $a = -\sqrt[3]{7}$ *and* $b = 0$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = 3\left(\sqrt[3]{7} - 3\right)$ *and* $q = 6\sqrt[3]{7} - 9$. *Following the steps in Algorithm 2, then*

$$ab + 6(a + b) + 9 = A = -6\sqrt[3]{7} + 9,$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = -4(-7) - 27 = 1 \implies \Delta = C = 1.$$

and

$$D = \frac{1}{2}(A + C) = \frac{1}{2}\left(-6\sqrt[3]{7} + 9 + 1\right) = -3\sqrt[3]{7} + 5,$$
$$E = \frac{1}{2}(A - C) = \frac{1}{2}\left(-6\sqrt[3]{7} + 9 - 1\right) = -3\sqrt[3]{7} + 4.$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}}.$$

The roots of $x^3 + \sqrt[3]{7}x^2 - 1 = 0$, *proved in [22], are* $\alpha = -\frac{2}{\sqrt[6]{7}}\sin\frac{2\pi}{7}$, $\beta = -\frac{2}{\sqrt[6]{7}}\sin\frac{4\pi}{7}$, *and* $\gamma = -\frac{2}{\sqrt[6]{7}}\sin\frac{8\pi}{7}$. *Then, by Theorem 1.1,*

$$\sqrt[3]{\sin\frac{2\pi}{7}} + \sqrt[3]{\sin\frac{4\pi}{7}} + \sqrt[3]{\sin\frac{8\pi}{7}} = \left(-\sqrt[18]{\frac{7}{64}}\right)\left(\sqrt[3]{-\sqrt[3]{7} + 6 + 3\left(\sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}}\right)}\right),$$

$$\frac{1}{\sqrt[3]{\sin\frac{2\pi}{7}}} + \frac{1}{\sqrt[3]{\sin\frac{4\pi}{7}}} + \frac{1}{\sqrt[3]{\sin\frac{8\pi}{7}}} = \left(-\sqrt[18]{\frac{64}{7}}\right)\left(\sqrt[3]{6 + 3\left(\sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}}\right)}\right).$$

Example 4.4. ([22]) *Let* $f(x) = x^3 - \sqrt[3]{7}x^2 - \sqrt[3]{49}x - 1 = 0$. *Denote* $a = \sqrt[3]{7}$ *and* $b = -\sqrt[3]{49}$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = -3\left(\sqrt[3]{7} - \sqrt[3]{49} + 3\right)$ *and* $q = 6\sqrt[3]{7} - 6\sqrt[3]{49} + 2$. *Following the steps in Algorithm 2, then*

$$ab + 6(a + b) + 9 = A = \sqrt[3]{7}\left(-\sqrt[3]{49}\right) + 6\left(\sqrt[3]{7} - \sqrt[3]{49}\right) + 9 = 2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49}\right).$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 49 - 4(7 - 49) - 126 - 27 = 64$$
$$\implies \Delta = C = 8,$$

and

$$D = \frac{1}{2}(A + C) = \frac{1}{2}\left(2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49}\right) + 8\right) = 5 + 3\left(\sqrt[3]{7} - \sqrt[3]{49}\right),$$
$$E = \frac{1}{2}(A - C) = \frac{1}{2}\left(2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49}\right) - 8\right) = -3 + 3\left(\sqrt[3]{7} - \sqrt[3]{49}\right).$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{5 + 3\left(\sqrt[3]{7} - \sqrt[3]{49}\right)} + \sqrt[3]{3\left(\sqrt[3]{7} - \sqrt[3]{49}\right) - 3}.$$

The roots of $x^3 - \sqrt[3]{7}x^2 - \sqrt[3]{49}x - 1 = 0$, proved in [22], are $\alpha = -\frac{1}{\sqrt[6]{7}}\tan\frac{2\pi}{7}$, $\beta = -\frac{1}{\sqrt[6]{7}}\tan\frac{4\pi}{7}$, and $\gamma = -\frac{1}{\sqrt[6]{7}}\tan\frac{8\pi}{7}$. Then, by Theorem 1.1,

$$\sqrt[3]{\tan\frac{2\pi}{7}} + \sqrt[3]{\tan\frac{4\pi}{7}} + \sqrt[3]{\tan\frac{8\pi}{7}}$$
$$= \left(\sqrt[18]{7}\right)\sqrt[3]{-\sqrt[3]{7} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)},$$
$$\frac{1}{\sqrt[3]{\tan\frac{2\pi}{7}}} + \frac{1}{\sqrt[3]{\tan\frac{4\pi}{7}}} + \frac{1}{\sqrt[3]{\tan\frac{8\pi}{7}}}$$
$$= \left(-\frac{1}{\sqrt[18]{7}}\right)\sqrt[3]{-\sqrt[3]{49} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)}.$$

The following cubic equations are selected from Wang's preprint [21]. In his preprint, no Ramanujan-type identities and proof of roots were given. The proof of the roots are given in the Appendix. The authors complete the Ramanujan-type identities in the following examples.

Example 4.5. ([21]) *Let $f(x) = x^3 + 4x^2 + 3x - 1 = 0$. Denote $a = -4$ and $b = 3$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = -6$ and $q = 9$. Following the steps in Algorithm 2, then*

$$ab + 6(a+b) + 9 = A = -4(3) + 6(-4+3) + 9 = -9.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 49 \implies \Delta = C = 7.$$

and

$$D = \frac{1}{2}(A + C) = \frac{1}{2}(-9 + 7) = -1,$$
$$E = \frac{1}{2}(A - C) = \frac{1}{2}(-9 - 7) = -8.$$

Then

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{-1} + \sqrt[3]{-8} = -3.$$

The roots of $x^3 + 4x^2 + 3x - 1 = 0$, proved in the Appendix, are $\alpha = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)$, $\beta = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)$, and $\gamma = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)$. Then, by Theorem 1.1,

$$\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)} + \sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)} + \sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)} = \sqrt[3]{-4 + 6 + 3(-3)}$$
$$\implies \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}. \quad (20)$$

$$\frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)}} + \frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)}} + \frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)}} = \sqrt[3]{3 + 6 + 3(-3)}$$
$$\implies \csc\left(\frac{2\pi}{7}\right) + \csc\left(\frac{4\pi}{7}\right) + \csc\left(\frac{8\pi}{7}\right) = 0. \quad (21)$$

Example 4.6. ([21]) *Let* $f(x) = x^3 + 4x^2 + 3x - 1 = 0$. *Denote* $a = -4$ *and* $b = 3$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = -6$ *and* $q = 9$. *From the previous example, we have* $t = -3$. *The roots of* $x^3 + 4x^2 + 3x - 1 = 0$, *proved in the Appendix, are* $\alpha = \frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}$, $\beta = \frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}$, *and* $\gamma = \frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}$. *Then, by Theorem 1.1,*

$$\sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} = -\sqrt[3]{7}, \tag{22}$$

$$\sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} = 0. \tag{23}$$

Example 4.7. ([21]) *Let* $f(x) = x^3 + 46x^2 + 3x - 1 = 0$. *Denote* $a = -46$ *and* $b = 3$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = 120$ *and* $q = 387$. *Following the steps in Algorithm 2, then*

$$ab + 6(a + b) + 9 = A = -46(3) + 6(-46 + 3) + 9 = -387.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 405769 \implies \Delta = C = 637,$$

and

$$\begin{aligned} D &= \frac{1}{2}(A + C) = \frac{1}{2}(-387 + 637) = 125, \\ E &= \frac{1}{2}(A - C) = \frac{1}{2}(-387 - 637) = -512. \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{125} + \sqrt[3]{-512} = -3.$$

The roots of $x^3 + 46x^2 + 3x - 1 = 0$, *proved in the Appendix, are* $\alpha = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3$, $\beta = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3$, *and* $\gamma = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3$. *Then by Theorem 1.1,*

$$\begin{aligned} &\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3} \\ &= \sqrt[3]{-46 + 6 + 3(-3)} \\ \implies &\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)} + \frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)} + \frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)} = 2\sqrt{7}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3}} \\ &= \sqrt[3]{3 + 6 + 3(-3)} \\ \implies &\frac{\sin^2\left(\frac{4\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} + \frac{\sin^2\left(\frac{8\pi}{7}\right)}{\sin\left(\frac{4\pi}{7}\right)} + \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{8\pi}{7}\right)} = 0. \end{aligned} \tag{25}$$

Example 4.8. ([21]) *Let* $f(x) = x^3 - 3x^2 - 46x - 1 = 0$*. Denote* $a = 3$ *and* $b = -46$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 120$ *and* $q = 387$*. From the last example,*

$$t = -3.$$

The roots of $x^3 - 3x^2 - 46x - 1 = 0$*, proved in the Appendix, are* $\alpha = 2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)$*,* $\beta = 2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)$*, and* $\gamma = 2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)$*. Then by Theorem 1.1,*

$$\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)} + \sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)} + \sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)} = \sqrt[3]{3+6+3(-3)}$$

$$\Longrightarrow \sqrt[3]{\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} = 0, \tag{26}$$

$$\frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)}} = \sqrt[3]{-46+6+3(-3)}$$

$$\Longrightarrow \sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos^4\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos^4\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos^4\left(\frac{8\pi}{7}\right)}} = -2\sqrt[3]{49}. \tag{27}$$

Example 4.9. ([21]) *Let* $f(x) = x^3 + 186x^2 + 3x - 1 = 0$*. Denote* $a = -186$ *and* $b = 3$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 540$ *and* $q = 1647$*. Following the steps in Algorithm 2, then*

$$ab + 6(a+b) + 9 = A = -186(3) + 6(-186+3) + 9 = -1647.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3+b^3) + 18ab - 27 = 26040609 \Longrightarrow \Delta = C = 5103.$$

and

$$D = \frac{1}{2}(A+C) = \frac{1}{2}(-1647+5103) = 1728,$$
$$E = \frac{1}{2}(A-C) = \frac{1}{2}(-1647-5103) = 3375.$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{1728} + \sqrt[3]{3375} = 12 - 15 = -3.$$

The roots of $x^3 + 186x^2 + 3x - 1 = 0$*, proved in the Appendix, are* $\alpha = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3$*,* $\beta = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3$*, and* $\gamma = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3$*. Then by Theorem 1.1,*

$$\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3}+\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3}+\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3}$$

$$= \sqrt[3]{-186+6+3(-3)}$$

$$\implies \frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}+\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}+\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)} = -12, \tag{28}$$

$$\frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3}}+\frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3}}+\frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3}}$$

$$= \sqrt[3]{3+6+3(-3)}$$

$$\implies \frac{\sin^3\left(\frac{4\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)}+\frac{\sin^3\left(\frac{8\pi}{7}\right)}{\sin\left(\frac{4\pi}{7}\right)}+\frac{\sin^3\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{8\pi}{7}\right)} = 0. \tag{29}$$

Example 4.10. ([21]) *Let $f(x) = x^3 - 3x^2 - 186x - 1 = 0$. Denote $a = 3$ and $b = -186$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = 540$ and $q = 1647$. From the last example*

$$t = -3.$$

The roots of $x^3 - 3x^2 - 186x - 1 = 0$, proved in the Appendix, are $\alpha = 2^3\left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}\right)$, $\beta = 2^3\left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)}\right)$, and $\gamma = 2^3\left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)}\right)$. Then by Theorem 1.1,

$$\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}\right)}+\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)}\right)}+\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)}\right)} = \sqrt{3+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}}+\sqrt[3]{\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)}}+\sqrt[3]{\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)}} = 0, \tag{30}$$

$$\frac{1}{\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}\right)}}+\frac{1}{\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)}\right)}}+\frac{1}{\sqrt[3]{2^3\left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)}\right)}} = \sqrt[3]{-186+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^2\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}}+\sqrt[3]{\frac{\cos^2\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}}+\sqrt[3]{\frac{\cos^2\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}} = -6\sqrt[3]{7}. \tag{31}$$

Example 4.11. ([21]) *Let $f(x) = x^3-3x^2-1588864x-1 = 0$. Denote $a = 3$ and $b = -1588864$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = 4766574$ and $q = 14299359$. Following the steps in Algorithm 2, then*

$$ab + 6(a + b) + 9 = A = 3(-1588864) + 6(a - 1588864) + 9 = -14299359.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3+b^3) + 18ab - 27 = 16044300277913161849$$
$$\implies \Delta = C = 4005533707,$$

and

$$D = \frac{1}{2}(A+C) = \frac{1}{2}(-14299359 + 4005533707) = 1995617174,$$
$$E = \frac{1}{2}(A-C) = \frac{1}{2}(-14299359 - 4005533707) = -2009916533.$$

Then

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{1995617174} + \sqrt[3]{-2009916533} = -3.$$

The roots of $x^3 - 3x^2 - 1588864x - 1 = 0$, proved in the Appendix, are $\alpha = 2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)$, $\beta = 2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)$, and $\gamma = 2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)$. Then by Theorem 1.1,

$$\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)} + \sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)} + \sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)} = \sqrt[3]{3+6+3(-3)}$$
$$\implies \sqrt[3]{\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}} = 0, \tag{32}$$

$$\frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)}} = \sqrt[3]{-1588864+6+3(-3)}$$
$$\implies \sqrt[3]{\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^{14}\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^{14}\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^{14}\left(\frac{8\pi}{7}\right)}} = -488\sqrt[3]{7}. \tag{33}$$

4.2 Equation $x^3 - ax^2 + bx - 1 = 0, a + b + 3 = 0$

In this section, we give examples on the cubic equation $x^3 - ax^2 + bx - 1 = 0$ with the condition $a + b + 3 = 0$. This condition has been heavily studied in literature, such as [2, 7, 16].

Example 4.12. *Let $f(x) = x^3 + x^2 - 2x - 1 = 0$. Denote $a = -1$ and $b = -2$. The associated Ramanujan equation is $t^3 - 7 = 0$. Hence, $t = -\sqrt[3]{7}$. The roots of $x^3 + x^2 - 2x - 1 = 0$ are $\alpha = 2\cos\frac{2\pi}{7}$, $\beta = 2\cos\frac{4\pi}{7}$, and $\gamma = 2\cos\frac{8\pi}{7}$* [11]. *By Theorem 1.1, we have the following Ramanujan-type identities:*

$$\sqrt[3]{2\cos\frac{2\pi}{7}} + \sqrt[3]{2\cos\frac{4\pi}{7}} + \sqrt[3]{2\cos\frac{8\pi}{7}} = \sqrt[3]{-1+6-3\sqrt[3]{7}}$$
$$\implies \sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}},$$

and

$$\frac{1}{\sqrt[3]{2\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{2\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{2\cos\frac{8\pi}{7}}}=\sqrt[3]{-2+6-3\sqrt[3]{7}}$$
$$\implies \frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}.$$

Example 4.13. *Let $f(x)=x^3+\frac{3}{2}x^2-\frac{3}{2}x-1=0$. Denote $a=-\frac{3}{2}$ and $b=-\frac{3}{2}$, i.e., $a+b+3=0$. By Corollary 2.2, $t=\sqrt[3]{\frac{9}{4}-9}=-\frac{3}{\sqrt[3]{4}}$.*

Using rational roots theorem, the roots of $f(x)=0$ are $\alpha=-2$,$\beta=-\frac{1}{2}$, and $\gamma=1$. Thus,

$$\sqrt[3]{-2}+\sqrt[3]{-\frac{1}{2}}+\sqrt[3]{1}=\sqrt[3]{-\frac{3}{2}+6-\frac{9}{\sqrt[3]{4}}}$$
$$\implies \sqrt[3]{2}-\sqrt[3]{4}-1=\sqrt[3]{9-\frac{18}{\sqrt[3]{4}}}.$$

Example 4.14. *Let $f(x)=x^3-\frac{3}{4}x+\frac{1}{8}=0$. It is known that the roots are $\alpha=\cos\frac{2\pi}{9}$, $\beta=\cos\frac{4\pi}{9}$, and $\gamma=\cos\frac{8\pi}{9}$* [11]. *Apply the transformation $x=-\frac{1}{2}y$, then we obtain*

$$-\frac{1}{8}y^3+\frac{3}{4}\cdot\frac{1}{2}y+\frac{1}{8}=0 \implies g(y)=y^3-3y-1=0.$$

Denote $a=0$ and $b=-3$, i.e., $a+b+3=0$. Then by Corollary 2.2, we have $t=\sqrt[3]{-9}$. Using the transformation, the roots of $g(y)=0$ are $y=-2\cos\frac{2\pi}{9},\ -2\cos\frac{4\pi}{9}$, and $-2\cos\frac{8\pi}{9}$. Thus,

$$\sqrt[3]{-2\cos\frac{2\pi}{9}}+\sqrt[3]{-2\cos\frac{4\pi}{9}}+\sqrt[3]{-2\cos\frac{8\pi}{9}}=\sqrt[3]{6-3\sqrt[3]{9}}$$
$$\implies \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=-\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}}, \qquad \textit{and}$$
$$\frac{1}{\sqrt[3]{-2\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{8\pi}{9}}}=\sqrt[3]{3-3\sqrt[3]{9}}$$
$$\implies \frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}}=-\sqrt[3]{6-6\sqrt[3]{9}}.$$

5 Analysis of the algorithms

The first step of Algorithm 1 is to find the roots of the cubic equation. To obtain Ramanujan-type identities, the roots generally involve trigonometric functions. Hence, Steps 2–6 involve trigonometric identities to simplify to a number. As a consequence, the computational complexity of this method is slow.

In our algorithm, only arithmetic calculations are involved. Since no trigonometric identities are involved in solving for t, the computational complexity is faster than that of Algorithm 1. Assume a and b have n number of digits. The complexity to compute Algorithm 2 relies solely on the time complexity of the chosen multiplication algorithm, assuming that Newton's method would be used to find the roots of the polynomial. Newton's method is currently the optimal algorithm to find roots, which has a time complexity is $O(M(n))$ where $M(n)$ is the complexity of the chosen multiplication algorithm. Our algorithm is self-converging and can provide all zeros for an algebraic cubic equation [9]. Schoolbook long multiplication being the worst and having $O(n^2)$. However, if the Harvey–Hoeven algorithm is used, then the time complexity would drop to being of the order $O(n \log n)$.

6 Conclusion

In this paper, two results were given to reduce the computational complexity of obtaining Wang's Ramanujan-type identities. Since Wang's method relied heavily on using the roots of the cubic equation and trigonometric identities for simplification, computational efficiency is reduced tremendously. Our method using Liao et al.'s method on the associated Ramanujan equation of the cubic equation results in a vast improvement in efficiency due to relying only on arithmetic calculations and reducing the total amount of computations required.

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Appendix

The trigonometric identities involving $\theta = \frac{\pi}{7}$ are obtained from heptagonal triangles. The heptagonal triangles have been studied in [1, 3, 4, 12, 19, 20, 26]. A heptagonal triangle is an obtuse scalene triangle whose vertices coincide with the first, second, and fourth vertices of a regular heptagon. Its angles have measures θ, 2θ, 4θ. Let $a < b < c$ be the sides of the heptagonal triangle and let R be the circumradius. Note. $\sin\theta = -\sin 8\theta$, $\sin 3\theta = \sin 4\theta$, $\cos\theta = -\cos 8\theta$, and $\cos 3\theta = -\cos 4\theta$. The following identities are selected from Bankoff [1]:

$$\sin\theta\sin 2\theta\sin 4\theta = \frac{\sqrt{7}}{8},$$
$$\cos\theta\cos 2\theta\cos 4\theta = -\frac{1}{8},$$
$$\cos 2\theta + \cos 4\theta + \cos 8\theta = -\frac{1}{2}$$

Some special identities include [3, 4, 12, 19]:

$$\frac{\sin 2\theta}{\sin\theta} - \frac{\sin 3\theta}{\sin 2\theta} + \frac{\sin\theta}{\sin 3\theta} = 1,$$
$$\frac{\sin\theta}{\sin 2\theta} - \frac{\sin 2\theta}{\sin 3\theta} + \frac{\sin 3\theta}{\sin\theta} = 2,$$
$$\frac{\sin^2\theta}{\sin 3\theta} - \frac{\sin^2 2\theta}{\sin\theta} + \frac{\sin^2 3\theta}{\sin 2\theta} = 0,$$
$$\frac{\sin 2\theta}{\sin^4\theta} - \frac{\sin\theta}{\sin^4 3\theta} + \frac{\sin 3\theta}{\sin^4 2\theta} = \frac{64\sqrt{7}}{7},$$
$$\frac{\sin^4 3\theta}{\sin\theta} - \frac{\sin^4\theta}{\sin 2\theta} - \frac{\sin^4 2\theta}{\sin 3\theta} = \frac{5\sqrt{7}}{8},$$
$$\frac{\sin^7 2\theta}{\sin^7\theta} - \frac{\sin^7 3\theta}{\sin^7 2\theta} + \frac{\sin^7\theta}{\sin^7 3\theta} = 57,$$
$$\frac{\sin^7\theta}{\sin^7 2\theta} - \frac{\sin^7 2\theta}{\sin^7 3\theta} + \frac{\sin^7 3\theta}{\sin^7\theta} = 289,$$
$$\frac{\sin^3 3\theta}{\sin^6\theta} - \frac{\sin^3\theta}{\sin^6 2\theta} + \frac{\sin^3 2\theta}{\sin^6 3\theta} = \frac{368}{\sqrt{7}},$$
$$\frac{\sin 2\theta}{\sin^2 3\theta} - \frac{\sin\theta}{\sin^2 2\theta} + \frac{\sin 3\theta}{\sin^2\theta} = 2\sqrt{7}.$$

In [19], Wang gave a method to obtain heptagonal identities involving sums of sine powers.

Proposition 1. ([19]) *Let* $\theta = \frac{\pi}{7}$. *Define* $S(n) = \sin^n(2\theta) + \sin^n(4\theta) + \sin^n(8\theta)$, *where* n *is a integer. Then* $S(n)$ *satisfies the recurrence relation:*

$$S(n) = \frac{\sqrt{7}}{2} S(n-1) - \frac{\sqrt{7}}{8} S(n-3).$$

Wang also provided values of $S(n)$ for $n = 1, \ldots, 20$:

$\boldsymbol{n}$	0	1	2	3	4	5	6
$\boldsymbol{S(n)}$	3	$\frac{\sqrt{7}}{2}$	$\frac{7}{2^2}$	$\frac{\sqrt{7}}{2}$	$\frac{21}{2^4}$	$\frac{7\sqrt{7}}{2^4}$	$\frac{35}{2^5}$
$\boldsymbol{S(-n)}$	3	0	2^3	$-\frac{2^3 \cdot 3\sqrt{7}}{7}$	2^5	$-\frac{2^5 \cdot 5\sqrt{7}}{7}$	$\frac{2^6 \cdot 17}{7}$

$\boldsymbol{n}$	7	8	9	10	11	12	13
$\boldsymbol{S(n)}$	$\frac{7^2\sqrt{7}}{2^7}$	$\frac{7^2 \cdot 5}{2^8}$	$\frac{7 \cdot 25\sqrt{7}}{2^9}$	$\frac{7^2 \cdot 9}{2^9}$	$\frac{7^2 \cdot 13\sqrt{7}}{2^{11}}$	$\frac{7^2 \cdot 33}{2^{11}}$	$\frac{7^2 \cdot 3\sqrt{7}}{2^9}$
$\boldsymbol{S(-n)}$	$-2^7\sqrt{7}$	$\frac{2^9 \cdot 11}{7}$	$-\frac{2^{10} \cdot 33\sqrt{7}}{7^2}$	$\frac{2^{10} \cdot 29}{7}$	$-\frac{2^{14} \cdot 11\sqrt{7}}{7^2}$	$\frac{2^{12} \cdot 269}{7^2}$	$-\frac{2^{13} \cdot 117\sqrt{7}}{7^2}$

$\boldsymbol{n}$	14	15	16	17	18	19	20
$\boldsymbol{S(n)}$	$\frac{7^4 \cdot 5}{2^{14}}$	$\frac{7^2 \cdot 179\sqrt{7}}{2^{15}}$	$\frac{7^3 \cdot 131}{2^{16}}$	$\frac{7^3 \cdot 3\sqrt{7}}{2^{12}}$	$\frac{7^3 \cdot 493}{2^{18}}$	$\frac{7^3 \cdot 181\sqrt{7}}{2^{18}}$	$\frac{7^5 \cdot 19}{2^{19}}$
$\boldsymbol{S(-n)}$	$\frac{2^{14} \cdot 51}{7}$	$-\frac{2^{21} \cdot 17\sqrt{7}}{7^3}$	$\frac{2^{17} \cdot 237}{7^2}$	$-\frac{2^{17} \cdot 1445\sqrt{7}}{7^3}$	$\frac{2^{19} \cdot 2203}{7^3}$	$-\frac{2^{19} \cdot 1919\sqrt{7}}{7^3}$	$\frac{2^{20} \cdot 5851}{7^3}$

Table 1. Values of $S(n)$ for $n = 1, \ldots, 20$.

We also provide his lemma here:

Lemma 6.1. ([19]) *Define* $W(m,n) = \sin^m(2\theta)\sin^n(4\theta) + \sin^m(4\theta)\sin^n(8\theta) + \sin^m(8\theta)\sin^n(2\theta)$, *where* m, n *are integers. Let* $P = \sin(2\theta)\sin(4\theta)\sin(8\theta) = -\frac{\sqrt{7}}{8}$. *Then*

$$\begin{aligned} W(m,n) + W(n,m) &= S(m)S(n) - S(m+n), \\ W(m,n)W(n,m) &= P^{m+n} S(-(m+n)) + P^m S(2n-m) + P^n S(2m-n). \end{aligned}$$

Similarly, Wang [20] also gave a method for sums of cosine powers.

Proposition 2. ([20]) *Let* $\theta = \frac{\pi}{7}$. *Define* $C(n) = \cos^n 2\theta + \cos^n 4\theta + \cos^n 8\theta$, *where* n *is an integer. Then* $C(n)$ *satisfies the recurrence relations*

$$\begin{aligned} C(n) &= -\frac{1}{2}C(n-1) + \frac{1}{2}C(n-2) + \frac{1}{8}C(n-3), \\ C(-n) &= -4C(-n+1) + 4C(-n+2) + 8C(-n+3). \end{aligned}$$

For $n = 0, \cdots, 10$, *we have*

$$C(n) = 3, -\frac{1}{2}, \frac{5}{4}, -\frac{1}{2}, \frac{13}{16}, -\frac{1}{2}, \frac{19}{32}, -\frac{57}{128}, \frac{117}{256}, -\frac{193}{512}, \frac{185}{512}, \cdots.$$

In addition, we have the following lemma:

Lemma 6.2. ([20]) *Define* $V(m,n) = \cos^m 2\theta \cos^n 4\theta + \cos^m 4\theta \cos^n 8\theta + \cos^m 8\theta \cos^n 2\theta$*, where* m, n *are integers. Then* $V(m,n)$ *satisfy the following recurrence relations*

$$V(m,n) = -\frac{1}{2}V(m-1,n) + \frac{1}{2}V(m-2,n) + \frac{1}{8}V(m-3,n),$$
$$V(m,n) = -\frac{1}{2}V(m,n-1) + \frac{1}{2}V(m,n-2) + \frac{1}{8}V(m,n-3).$$

The table for $V(m,n)$*,* $m,n = 1,\ldots,8$*:*

$m \backslash n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{16}$	$-\frac{9}{32}$	$\frac{3}{64}$	$-\frac{11}{4}$	$\frac{19}{256}$	$-\frac{15}{128}$
2	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{3}{64}$	$\frac{19}{256}$	$\frac{3}{512}$	$\frac{41}{1024}$
3	$-\frac{1}{2}$	$-\frac{9}{32}$	$-\frac{11}{64}$	$-\frac{15}{128}$	$-\frac{1}{16}$	$-\frac{25}{512}$	$-\frac{11}{512}$	$-\frac{11}{512}$
4	$\frac{3}{8}$	$\frac{17}{64}$	$\frac{5}{32}$	$\frac{13}{128}$	$\frac{31}{512}$	$\frac{41}{1024}$	$\frac{47}{2048}$	$\frac{33}{2048}$
5	$-\frac{25}{64}$	$-\frac{29}{128}$	$-\frac{37}{256}$	$-\frac{23}{256}$	$-\frac{57}{1024}$	$-\frac{9}{256}$	$-\frac{11}{512}$	$-\frac{113}{8912}$
6	$\frac{41}{128}$	$\frac{27}{128}$	$\frac{33}{256}$	$\frac{83}{1024}$	$\frac{103}{2048}$	$\frac{129}{4096}$	$\frac{5}{256}$	$\frac{201}{16384}$
7	$-\frac{79}{256}$	$-\frac{95}{512}$	$-\frac{15}{128}$	$-\frac{149}{2048}$	$-\frac{93}{2048}$	$-\frac{29}{1024}$	$-\frac{289}{16384}$	$-\frac{361}{32768}$
8	$\frac{17}{64}$	$\frac{87}{512}$	$\frac{215}{2048}$	$\frac{269}{4096}$	$\frac{335}{8192}$	$\frac{209}{8192}$	$\frac{521}{32768}$	$\frac{325}{32768}$

Table 2. Values of $V(m,n)$ for $m,n = 1,\ldots,8$.

We refer the reader to [1, 19] for the other heptagonal identities.

Proof of roots in Example 4.5

The roots of $x^3 + 4x^2 + 3x - 1 = 0$ are $\alpha = -\frac{8}{\sqrt{7}}\sin^3(2\theta)$, $\beta = -\frac{8}{\sqrt{7}}\sin^3(4\theta)$, and $\gamma = -\frac{8}{\sqrt{7}}\sin^3(8\theta)$.

Proof. Denote $\theta = \frac{\pi}{7}$. Apply Vieta's formula, the first part is

$$\begin{aligned}\alpha + \beta + \gamma &= -\frac{8}{\sqrt{7}}\sin^3(2\theta) - \frac{8}{\sqrt{7}}\sin^3(4\theta) - \frac{8}{\sqrt{7}}\sin^3(8\theta) \\ &= -\frac{8}{\sqrt{7}}\left(\sin^3(2\theta) + \sin^3(4\theta) + \sin^3(8\theta)\right) = -\frac{8}{\sqrt{7}}\left(\frac{\sqrt{7}}{2}\right) = -4,\end{aligned}$$

where $\sin^3(2\theta) + \sin^3(4\theta) + \sin^3(8\theta) = \frac{\sqrt{7}}{2}$ [19]. Next,

$$\begin{aligned}\alpha\beta + \beta\gamma + \gamma\alpha &= \frac{64}{7}\left(\sin^3(2\theta)\sin^3(4\theta) + \sin^3(4\theta)\sin^3(8\theta) + \sin^3(8\theta)\sin^3(2\theta)\right) \\ &= \frac{64}{7}\left(\frac{21}{64}\right) = 3,\end{aligned}$$

where $\sin^3(2\theta)\sin^3(4\theta)+\sin^3(4\theta)\sin^3(8\theta)+\sin^3(8\theta)\sin^3(2\theta)=W(3,3)=\frac{21}{64}$ [19]. Lastly,

$$\begin{aligned}\alpha\beta\gamma &= -\left(\frac{8}{\sqrt{7}}\right)^3 \sin^3(2\theta)\sin^3(4\theta)\sin^3(8\theta) = -\left(\frac{8}{\sqrt{7}}\right)^3\left(-\frac{7\sqrt{7}}{512}\right)\\ &= -\left(\frac{8}{\sqrt{7}}\right)^3 (\sin(2\theta)\sin(4\theta)\sin(8\theta))^3 = -\left(\frac{8}{\sqrt{7}}\right)^3\left(-\frac{\sqrt{7}}{8}\right)^3 = 1,\end{aligned}$$

where $\sin\theta\sin 2\theta\sin 4\theta = \frac{\sqrt{7}}{8} = -\sin 2\theta\sin 4\theta\sin 8\theta$ [1]. □

Proof of roots in Example 4.6

The roots of $x^3+4x^2+3x-1=0$ are $\alpha=\frac{\cos(2\theta)}{\cos(4\theta)}$, $\beta=\frac{\cos(4\theta)}{\cos(8\theta)}$, and $\gamma=\frac{\cos(8\theta)}{\cos(2\theta)}$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}\alpha+\beta+\gamma &= \frac{\cos(2\theta)}{\cos(4\theta)}+\frac{\cos(4\theta)}{\cos(8\theta)}+\frac{\cos(8\theta)}{\cos(2\theta)}\\ &= \frac{\cos^2 2\theta\cos 8\theta+\cos^2 4\theta\cos 2\theta+\cos^2 8\theta\cos 4\theta}{\cos 2\theta\cos 4\theta\cos 8\theta}\\ &= 8\left(\cos^2 2\theta\cos 8\theta+\cos^2 4\theta\cos 2\theta+\cos^2 8\theta\cos 4\theta\right) = 8\left(-\frac{1}{2}\right) = -4,\end{aligned}$$

where by Lemma 6.1. $(\cos^2 2\theta\cos 8\theta+\cos^2 4\theta\cos 2\theta+\cos^2 8\theta\cos 4\theta)=V(1,2)=-\frac{1}{2}$ Next,

$$\begin{aligned}\alpha\beta+\beta\gamma+\gamma\alpha &= \frac{\cos(2\theta)}{\cos(4\theta)}\frac{\cos(4\theta)}{\cos(8\theta)}+\frac{\cos(4\theta)}{\cos(8\theta)}\frac{\cos(8\theta)}{\cos(2\theta)}+\frac{\cos(8\theta)}{\cos(2\theta)}\frac{\cos(2\theta)}{\cos(4\theta)}\\ &= \frac{\cos^2 2\theta\cos 4\theta+\cos^2 4\theta\cos 8\theta+\cos^2 8\theta\cos 2\theta}{\cos 2\theta\cos 4\theta\cos 8\theta}\\ &= 8\left(\cos^2 2\theta\cos 4\theta+\cos^2 4\theta\cos 8\theta+\cos^2 8\theta\cos 2\theta\right)\\ &= 8\left(\frac{3}{8}\right) = 3,\end{aligned}$$

where by Lemma 6.1 $(\cos^2 2\theta\cos 4\theta+\cos^2 4\theta\cos 8\theta+\cos^2 8\theta\cos 2\theta)=V(2,1)=\frac{3}{8}$. Finally,

$$\alpha\beta\gamma = \frac{\cos(2\theta)}{\cos(4\theta)}\frac{\cos(4\theta)}{\cos(8\theta)}\frac{\cos(8\theta)}{\cos(2\theta)} = 1.$$

□

Proof of roots in Example 4.7

The roots of $x^3+46x^2+3x-1=0$ are $\alpha=-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(2\theta)}{\sin^2(4\theta)}\right)\right)^3$, $\beta=-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(4\theta)}{\sin^2(8\theta)}\right)\right)^3$, and $\gamma=-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(8\theta)}{\sin^2(2\theta)}\right)\right)^3$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}\alpha+\beta+\gamma &= -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(2\theta)}{\sin^2(4\theta)}\right)\right)^3-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(4\theta)}{\sin^2(8\theta)}\right)\right)^3-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin(8\theta)}{\sin^2(2\theta)}\right)\right)^3\\ &= -\frac{\sqrt{7}}{8}\left(\frac{\sin^3(2\theta)}{\sin^6(4\theta)}+\frac{\sin^3(4\theta)}{\sin^6(8\theta)}+\frac{\sin^3(8\theta)}{\sin^6(2\theta)}\right) = -\frac{\sqrt{7}}{8}\left(\frac{368}{\sqrt{7}}\right)\\ &= -46,\end{aligned}$$

where $\left(\frac{\sin^3(2\theta)}{\sin^6(4\theta)}+\frac{\sin^3(4\theta)}{\sin^6(8\theta)}+\frac{\sin^3(8\theta)}{\sin^6(2\theta)}\right)=\frac{368}{\sqrt{7}}$ [19]. Next,

$$\begin{aligned}&\alpha\beta+\beta\gamma+\gamma\alpha\\ &= \frac{7}{64}\left(\left(\frac{\sin(2\theta)}{\sin^2(4\theta)}\right)^3\left(\frac{\sin(4\theta)}{\sin^2(8\theta)}\right)^3+\left(\frac{\sin(4\theta)}{\sin^2(8\theta)}\right)^3\left(\frac{\sin(8\theta)}{\sin^2(2\theta)}\right)^3+\left(\frac{\sin(8\theta)}{\sin^2(2\theta)}\right)^3\left(\frac{\sin(2\theta)}{\sin^2(4\theta)}\right)^3\right)\\ &= \frac{7}{64}\left(\frac{\sin^9 2\theta\sin^3 4\theta+\sin^9 4\theta\sin^3 8\theta+\sin^9 8\theta\sin^3 2\theta}{\sin^6 2\theta\sin^6 4\theta\sin^6 8\theta}\right)\\ &= \frac{7}{64}\left(-\frac{8}{\sqrt{7}}\right)^6\left(\sin^9 2\theta\sin^3 4\theta+\sin^9 4\theta\sin^3 8\theta+\sin^9 8\theta\sin^3 2\theta\right)=\frac{4096}{49}\left(\frac{147}{4096}\right)=3,\end{aligned}$$

where by Lemma 6.2 $\left(\sin^9 2\theta\sin^3 4\theta+\sin^9 4\theta\sin^3 8\theta+\sin^9 8\theta\sin^3 2\theta\right)=W(9,3)=\frac{147}{4096}$. Finally,

$$\begin{aligned}\alpha\beta\gamma &= \left(-\frac{\sqrt[6]{7}}{2}\right)^9\left(\left(\frac{\sin(2\theta)}{\sin^2(4\theta)}\right)^3\left(\frac{\sin(4\theta)}{\sin^2(8\theta)}\right)^3\left(\frac{\sin(8\theta)}{\sin^2(2\theta)}\right)^3\right)\\ &= -\frac{7\sqrt{7}}{512}\left(\frac{1}{\sin(2\theta)\sin(4\theta)\sin(8\theta)}\right)^3=-\frac{7\sqrt{7}}{512}\left(-\frac{8}{\sqrt{7}}\right)^3=1.\end{aligned}$$

□

Proof of roots in Example 4.8

The roots of $x^3-3x^2-46x-1=0$ are $\alpha=2^3\left(\frac{\cos^4(2\theta)}{\cos(4\theta)}\right)$, $\beta=2^3\left(\frac{\cos^4(4\theta)}{\cos(8\theta)}\right)$, and $\gamma=2^3\left(\frac{\cos^4(8\theta)}{\cos(2\theta)}\right)$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}\alpha+\beta+\gamma &= 2^3\left(\frac{\cos^4(2\theta)}{\cos(4\theta)}\right)+2^3\left(\frac{\cos^4(4\theta)}{\cos(8\theta)}\right)+2^3\left(\frac{\cos^4(8\theta)}{\cos(2\theta)}\right)\\ &= 8\left(\frac{\cos^4(2\theta)}{\cos(4\theta)}+\frac{\cos^4(4\theta)}{\cos(8\theta)}+\frac{\cos^4(8\theta)}{\cos(2\theta)}\right)\\ &= 8\left(\frac{\cos^5 2\theta\cos 8\theta+\cos^5 4\theta\cos 2\theta+\cos^5 8\theta\cos 4\theta}{\cos 2\theta\cos 4\theta\cos 8\theta}\right)\\ &= 8(8)\left(\frac{3}{64}\right)=3,\end{aligned}$$

where by Lemma 6.1 $\cos^5 2\theta \cos 8\theta + \cos^5 4\theta \cos 2\theta + \cos^5 8\theta \cos 4\theta = V(1,5) = \frac{3}{64}$. Next,

$$\begin{aligned}
\alpha\beta + \beta\gamma + \gamma\alpha &= 2^6 \left(\frac{\cos^4(2\theta)}{\cos(4\theta)} \frac{\cos^4(4\theta)}{\cos(8\theta)} + \frac{\cos^4(4\theta)}{\cos(8\theta)} \frac{\cos^4(8\theta)}{\cos(2\theta)} + \frac{\cos^4(8\theta)}{\cos(2\theta)} \frac{\cos^4(2\theta)}{\cos(4\theta)} \right) \\
&= 64 \left(\frac{\cos^5 2\theta \cos^4 4\theta + \cos^5 4\theta \cos^4 8\theta + \cos^5 8\theta \cos^4 2\theta}{\cos 2\theta \cos 4\theta \cos 8\theta} \right) \\
&= 64\,(8) \left(-\frac{23}{256} \right) = -46,
\end{aligned}$$

where by Lemma 6.1 $\cos^5 2\theta \cos^4 4\theta + \cos^5 4\theta \cos^4 8\theta + \cos^5 8\theta \cos^4 2\theta = V(5,4) = -\frac{23}{256}$. Finally,

$$\begin{aligned}
\alpha\beta\gamma &= 2^9 \left(\frac{\cos^4(2\theta)}{\cos(4\theta)} \right) \left(\frac{\cos^4(4\theta)}{\cos(8\theta)} \right) \left(\frac{\cos^4(8\theta)}{\cos(2\theta)} \right) = 2^9 \left(\cos(2\theta) \cos(4\theta) \cos(8\theta) \right)^3 \\
&= 2^9 \left(\frac{1}{8} \right)^3 = 1.
\end{aligned}$$

□

Proof of roots in Example 4.9

The roots of $x^3 + 186x^2 + 3x - 1 = 0$ are $\alpha = \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(2\theta)}{\sin^3(4\theta)} \right) \right)^3$, $\beta = \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(4\theta)}{\sin^3(8\theta)} \right) \right)^3$, and $\gamma = \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(8\theta)}{\sin^3(2\theta)} \right) \right)^3$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}
\alpha + \beta + \gamma &= \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(2\theta)}{\sin^3(4\theta)} \right) \right)^3 + \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(4\theta)}{\sin^3(8\theta)} \right) \right)^3 + \left(\frac{\sqrt[3]{7}}{4} \left(\frac{\sin(8\theta)}{\sin^3(2\theta)} \right) \right)^3 \\
&= \frac{7}{64} \left(\frac{\sin^3(2\theta)}{\sin^9(4\theta)} + \frac{\sin^3(4\theta)}{\sin^9(8\theta)} + \frac{\sin^3(8\theta)}{\sin^9(2\theta)} \right) \\
&= \frac{7}{64} \left(\frac{\sin^{12} 2\theta \sin^9 8\theta + \sin^{12} 4\theta \sin^9 2\theta + \sin^{12} 8\theta \sin^9 4\theta}{(\sin 2\theta \sin 4\theta \sin 8\theta)^9} \right) \\
&= \frac{7}{64} \left(-\frac{8}{\sqrt{7}} \right)^9 \left(\frac{31899\sqrt{7}}{1048576} \right) = -186,
\end{aligned}$$

where by Lemma 6.2 $\sin^{12} 2\theta \sin^9 8\theta + \sin^{12} 4\theta \sin^9 2\theta + \sin^{12} 8\theta \sin^9 4\theta = W(9,12) = \frac{31899\sqrt{7}}{1048576}$. Next,

$$\begin{aligned}
&\alpha\beta + \beta\gamma + \gamma\alpha \\
&\quad = \frac{49}{4096} \left(\left(\frac{\sin(2\theta)}{\sin^3(4\theta)} \right)^3 \left(\frac{\sin(4\theta)}{\sin^3(8\theta)} \right)^3 + \left(\frac{\sin(4\theta)}{\sin^3(8\theta)} \right)^3 \left(\frac{\sin(8\theta)}{\sin^3(2\theta)} \right)^3 + \left(\frac{\sin(8\theta)}{\sin^3(2\theta)} \right)^3 \left(\frac{\sin(2\theta)}{\sin^3(4\theta)} \right)^3 \right) \\
&\quad = \frac{49}{4096} \left(\frac{\sin^3(2\theta) \sin^3(4\theta)}{\sin^9(4\theta) \sin^9(8\theta)} + \frac{\sin^3(4\theta) \sin^3(8\theta)}{\sin^9(8\theta) \sin^9(2\theta)} + \frac{\sin^3(8\theta) \sin^3(2\theta)}{\sin^9(2\theta) \sin^9(4\theta)} \right) \\
&\quad = \frac{49}{4096} \left(\frac{\sin^{12} 2\theta \sin^3 4\theta + \sin^{12} 4\theta \sin^3 8\theta + \sin^{12} 8\theta \sin^3 2\theta}{(\sin 2\theta \sin 4\theta \sin 8\theta)^9} \right) \\
&\quad = \frac{49}{4096} \left(-\frac{8}{\sqrt{7}} \right)^9 \left(-\frac{147\sqrt{7}}{32768} \right) = 3,
\end{aligned}$$

where by Lemma 6.2 $\sin^{12} 2\theta \sin^3 4\theta + \sin^{12} 4\theta \sin^3 8\theta + \sin^{12} 8\theta \sin^3 2\theta = W(12, 3) = -\frac{147\sqrt{7}}{32768}$. Finally,

$$\begin{aligned}
\alpha\beta\gamma &= \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin(2\theta)}{\sin^3(4\theta)}\right)\right)^3 \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin(4\theta)}{\sin^3(8\theta)}\right)\right)^3 \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin(8\theta)}{\sin^3(2\theta)}\right)\right)^3 \\
&= \frac{7^3}{2^{18}}\left(\frac{\sin(2\theta)}{\sin^3(4\theta)}\right)^3 \left(\frac{\sin(4\theta)}{\sin^3(8\theta)}\right)^3 \left(\frac{\sin(8\theta)}{\sin^3(2\theta)}\right)^3 \\
&= \frac{7^3}{2^{18}}\frac{1}{(\sin 2\theta \sin 4\theta \sin 8\theta)^6} = \frac{7^3}{2^{18}}\left(-\frac{8}{\sqrt{7}}\right)^6 = 1.
\end{aligned}$$

□

Proof of roots in Example 4.10

The roots of $x^3 - 3x^2 - 186x - 1 = 0$ are $\alpha = 2^3\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)}\right)$, $\beta = 2^3\left(\frac{\cos^5(4\theta)}{\cos^2(8\theta)}\right)$, and $\gamma = 2^3\left(\frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right)$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}
\alpha + \beta + \gamma &= 2^3\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)}\right) + 2^3\left(\frac{\cos^5(4\theta)}{\cos^2(8\theta)}\right) + 2^3\left(\frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right) \\
&= 2^3\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)} + \frac{\cos^5(4\theta)}{\cos^2(8\theta)} + \frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right) \\
&= 8\left(\frac{\cos^7 2\theta \cos^2 8\theta + \cos^7 4\theta \cos^2 2\theta + \cos^7 8\theta \cos^2 4\theta}{\cos^2 2\theta \cos^2 4\theta \cos^2 8\theta}\right) \\
&= 8\,(64)\left(\frac{3}{512}\right) = 3,
\end{aligned}$$

where by Lemma 6.1 $\cos^7 2\theta \cos^2 8\theta + \cos^7 4\theta \cos^2 2\theta + \cos^7 8\theta \cos^2 4\theta = V(2, 7) = \frac{3}{512}$. Next,

$$\begin{aligned}
&\alpha\beta + \beta\gamma + \gamma\alpha \\
&= 2^6\left(\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)}\right)\left(\frac{\cos^5(4\theta)}{\cos^2(8\theta)}\right) + \left(\frac{\cos^5(4\theta)}{\cos^2(8\theta)}\right)\left(\frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right) + \left(\frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right)\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)}\right)\right) \\
&= 2^6\left(\frac{\cos^7 2\theta \cos^5 4\theta + \cos^7 4\theta \cos^5 8\theta + \cos^7 8\theta \cos^5 2\theta}{\cos^2 2\theta \cos^2 4\theta \cos^2 8\theta}\right) \\
&= 2^6\,(64)\left(-\frac{93}{2048}\right) = -186,
\end{aligned}$$

where by Lemma 6.1 $\cos^7 2\theta \cos^5 4\theta + \cos^7 4\theta \cos^5 8\theta + \cos^7 8\theta \cos^5 2\theta = V(7, 5) = -\frac{93}{2048}$. Finally,

$$\begin{aligned}
\alpha\beta\gamma &= 2^9\left(\frac{\cos^5(2\theta)}{\cos^2(4\theta)}\right)\left(\frac{\cos^5(4\theta)}{\cos^2(8\theta)}\right)\left(\frac{\cos^5(8\theta)}{\cos^2(2\theta)}\right) \\
&= 2^9(\cos 2\theta \cos 4\theta \cos 8\theta)^3 = 2^9\left(\frac{1}{8}\right)^3 = 1.
\end{aligned}$$

□

Proof of roots in Example 4.11

The roots of $x^3 - 3x^2 - 1588864x - 1 = 0$ are $\alpha = 2^9\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}\right)$, $\beta = 2^9\left(\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}\right)$, and $\gamma = 2^9\left(\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)$.

Proof. Apply Vieta's formula, the first part is

$$\begin{aligned}\alpha+\beta+\gamma &= 2^9\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}\right)+2^9\left(\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}\right)+2^9\left(\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)\\ &= 2^9\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}+\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}+\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)\\ &= 2^9\left(\frac{\cos^{19}2\theta\cos^5 8\theta+\cos^{19}4\theta\cos^5 2\theta+\cos^{19}8\theta\cos^5 4\theta}{\cos^5 2\theta\cos^5 4\theta\cos^5 8\theta}\right)\\ &= 2^9(8)^5\left(\frac{3}{2^{24}}\right)=3,\end{aligned}$$

where by Lemma 6.1 $\cos^{19}2\theta\cos^5 8\theta+\cos^{19}4\theta\cos^5 2\theta+\cos^{19}8\theta\cos^5 4\theta = V(5,19) = \frac{3}{2^{24}}$. Next,

$$\begin{aligned}&\alpha\beta+\beta\gamma+\gamma\alpha\\ &= 2^{18}\left(\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}\right)\left(\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}\right)+\left(\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}\right)\left(\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)+\left(\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}\right)\right)\\ &= 2^{18}\left(\frac{\cos^{19}(2\theta)\cos^{14}(4\theta)+\cos^{19}(4\theta)\cos^{14}(8\theta)+\cos^{19}(8\theta)\cos^{14}(2\theta)}{\cos^5 2\theta\cos^5 4\theta\cos^5 8\theta}\right)\\ &= 2^{18}(8)^5\left(-\frac{12413}{2^{26}}\right)=-1588864,\end{aligned}$$

where by Lemma 6.1 $\cos^{19}2\theta\cos^{14}4\theta+\cos^{19}4\theta\cos^{14}8\theta+\cos^{19}8\theta\cos^{14}2\theta = V(19,14) = -\frac{12413}{2^{26}}$. Finally,

$$\begin{aligned}\alpha\beta\gamma &= 2^{27}\left(\frac{\cos^{14}(2\theta)}{\cos^5(4\theta)}\right)\left(\frac{\cos^{14}(4\theta)}{\cos^5(8\theta)}\right)\left(\frac{\cos^{14}(8\theta)}{\cos^5(2\theta)}\right)\\ &= 2^{27}(8)^5\left(\frac{1}{8}\right)^{14}=1.\end{aligned}$$

□