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Binomial sums with k-Jacobsthal and k-Jacobsthal–Lucas numbers

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Abstract: In this paper, we derive some important identities involving k-Jacobsthal and k-Jacobsthal–Lucas numbers. Moreover, we use multinomial theorem to obtain distinct binomial sums of k-Jacobsthal and k-Jacobsthal–Lucas numbers.

Keywords: Jacobsthal number, Jacobsthal–Lucas number, k-Jacobsthal number, k-Jacobsthal–Lucas number.
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1 Introduction

The Fibonacci number is generalized by varying recurrence relations, initial conditions or both of them. The Jacobsthal and Jacobsthal–Lucas numbers are well-known examples of second-order generalized Fibonacci numbers. In the last few years, many authors have investigated Jacobsthal and Jacobsthal–Lucas numbers. In [8], Köken and Bozkurt derived fundamental identities and Binet-like formula for the Jacobsthal and Jacobsthal–Lucas numbers using matrices. Čerin [3] discovered sums of squares and products of Jacobsthal numbers.

The k-Jacobsthal and k-Jacobsthal–Lucas number is one more example of a generalized Fibonacci number. In recent years, many authors have studied these numbers. For example, see [1, 6, 7, 9–14] and the references cited therein.

In the year 1996 A. F. Horadam [5] first studied the Jacobsthal numbers. We reproduce following Definitions 1.1 and 1.2 from [5].

Definition 1.1. *The Jacobsthal numbers* $\Im_n$ *are defined by the recurrence relation*

$$\Im_{n+2} = \Im_{n+1} + 2\Im_n,$$

for $n \geq 0$ *with the initial conditions* $\Im_0 = 0$ *and* $\Im_1 = 1$.

Definition 1.2. *The Jacobsthal–Lucas numbers* $\wp_n$ *are given by the recurrence relation*

$$\wp_{n+2} = \wp_{n+1} + 2\wp_n,$$

for $n \geq 0$ *with the initial conditions* $\wp_0 = 2$ *and* $\wp_1 = 1$.

The Binet formula of Jacobsthal and Jacobsthal–Lucas numbers are given by

$$\wp_n = a^n + b^n \tag{1}$$

and

$$\Im_n = \frac{a^n - b^n}{a - b}, \tag{2}$$

where $a = 2$ and $b = 1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

Later, in 2015, Uygun and Eldogan, [10, 11, 13, 14] defined k-Jacobsthal and k-Jacobsthal–Lucas numbers and investigated various properties of these numbers. We rewrite following Definitions 1.3 and 1.4 from [10].

Definition 1.3. *The* k*-Jacobsthal numbers are given by the recurrence relation*

$$\Im_{k,n+1} = k\Im_{k,n} + 2\Im_{k,n-1},$$

for $n \geq 2$ *with the initial conditions* $\Im_{k,0} = 0$ *and* $\Im_{k,1} = 1$.

Definition 1.4. *The* k*-Jacobsthal–Lucas numbers by the recurrence relation*

$$\wp_{k,n+1} = k\wp_{k,n} + 2\wp_{k,n-1},$$

for $n \geq 2$ *with the initial conditions* $\wp_{k,0} = 2$ *and* $\wp_{k,1} = k$.

Binet Formula of k-Jacobsthal ($\Im_{k,n}$) and k-Jacobsthal–Lucas ($\wp_{k,n}$) numbers are given by

$$\wp_{k,n} = \eta_1^n + \eta_2^n \tag{3}$$

and

$$\Im_{k,n} = \frac{\eta_1^n - \eta_2^n}{\eta_1 - \eta_2}, \tag{4}$$

where $\eta_1 = \dfrac{k + \sqrt{k^2 + 8}}{2}$ and $\eta_2 = \dfrac{k - \sqrt{k^2 + 8}}{2}$ are the roots of the characteristic equation $x^2 - kx - 2 = 0$.

The characteristic roots η_1 and η_1 appeared in the Definitions 1.3 and 1.4 satisfy the following relations

$$\eta_1 - \eta_2 = \sqrt{k^2+8} = \sqrt{\delta}, \tag{5}$$

$$\eta_1 + \eta_2 = k, \tag{6}$$

$$\eta_1\eta_2 = -2. \tag{7}$$

$$\eta_1^2 = k\eta_1 + 2 \tag{8}$$

$$\eta_2^2 = k\eta_2 + 2 \tag{9}$$

The k-Jacobsthal and k-Jacobsthal–Lucas numbers are particular examples of Jacobsthal and Jacobsthal–Lucas numbers, respectively. The Jacobsthal numbers and the Jacobsthal–Lucas numbers are the origins of many fascinating properties. A complementary interpretation exists for k-Jacobsthal numbers and kJacobsthal–Lucas numbers. Many authors studied these numbers, see [1,6,7,9–14] and the references cited therein. Some of these are listed below

Lemma 1.1. *Let* $n, m \in \mathbb{Z}^+$. *Then*

1. $\Im_{k,n+1} + 2\Im_{k,n-1} = \wp_{k,n},$ (10)
2. $\Im_{k,n+1} + 2\wp_{k,n-1} = \delta\Im_{k,n},$ (11)
3. $2\Im_{k,m+n} = \Im_{k,m}\wp_{k,n} + \Im_{k,n}\wp_{k,m},$ (12)
4. $2\Im_{k,m-n} = \big(-1\big)^n\big(\Im_{k,m}\wp_{k,n} - \Im_{k,n}\wp_{k,m}\big),$ (13)
5. $\Im_{k,n-1}\Im_{k,n+1} - \Im_{k,n}^2 = -(-2)^{n-1},$ (14)
6. $\wp_{k,n-1}\wp_{k,n+1} - \wp_{k,n}^2 = (-2)^{n-1}\delta,$ (15)
7. $\Im_{k,n}\wp_{k,n} = \Im_{k,2n},$ (16)
8. $\wp_{k,n}^2 = \delta\Im_{k,n}^2 + 4(-2)^n,$ (17)
9. $\wp_{k,n} = k\Im_{k,2n} + 4\Im_{k,n-1},$ (18)
10. $k\Im_{k,n} + \wp_{k,n} = 2\Im_{k,n+1},$ (19)
11. $\delta\Im_{k,n} + k\wp_{k,n} = 2\wp_{k,n+1},$ (20)

We are influenced by the idea of Carlitz and Ferns [2], and motivated by the work of Zhang [15] to establish different binomial sums of k-Jacobsthal and k-Jacobsthal–Lucas numbers, see also [4].

2 Binomial sums with k-Jacobsthal and k-Jacobsthal–Lucas numbers

In this section, we establish some more binomial sums for k-Jacobsthal and k-Jacobsthal–Lucas numbers. The following Lemma 2.1 plays a key role in proving the Theorems 2.2 to 2.4.

Lemma 2.1. *Let* $u = \eta_1$ *or* η_2. *Then*

$$2^{(2^{n+1}+1)} + ku2^{(2^{n+1})} + u^{2(2^{n+1}+1)} = \wp_{k,2^{n+1}}u^{2(2^n+1)}. \tag{21}$$

Proof. Let $u = \eta_1$. Using the Binet formula of $\wp_{k,n}$, we have

$$\begin{aligned}
\wp_{k,2^{n+1}}\eta_1^{2(2^n+1)} &= \big(\eta_1^{2^{n+1}} + \eta_2^{2^{n+1}}\big)\eta_1^{2(2^n+1)} \\
&= \eta_1^{2^{n+1}}\eta_1^{2^{n+1}+2} + \eta_2^{2^{n+1}}\eta_1^{2^{n+1}+2} \\
&= \big(\eta_1^{2^{n+1}}\eta_1^{2^{n+1}}\big)\eta_1^2 + \big(\eta_2^{2^{n+1}}\eta_1^{2^{n+1}}\big)\eta_1^2 \\
&= \eta_1^2\big(\eta_1^{2^{n+1}+2^{n+1}} + (-2)^{2^{n+1}}\big) \\
&= \eta_1^2\big(\eta_1^{2^{n+2}} + 2^{(2^{n+1})}\big) \\
&= \big(\eta_1^2 2^{(2^{n+1})} + \eta_1^2\eta_1^{2^{n+2}}\big) \\
&= \big(\eta_1^2 2^{(2^{n+1})} + \eta_1^{2+2^{n+2}}\big) \\
&= (2 + k\eta_1)2^{(2^{n+1})} + \eta_1^{2(2^{n+1}+1)} \\
&= 2^{(2^{n+1}+1)} + k\eta_1 2^{(2^{n+1})} + \eta_1^{2(2^{n+1}+1)}
\end{aligned}$$

Similarly, we can prove the result for $u = \eta_2$. □

Theorem 2.1. *Let $n, t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

$$(1)\quad \Im_{k,t+2^{n+1}+2}\wp_{k,2^{n+1}} = 2^{(2^{n+1}+1)}\Im_{k,t} + k2^{(2^{n+1})}\Im_{k,t+1} + \Im_{k,t+2^{n+2}+2}, \tag{22}$$

$$(2)\quad \wp_{k,t+2^{n+1}+2}\wp_{k,2^{n+1}} = 2^{(2^{n+1}+1)}\wp_{k,t} + k2^{(2^{n+1})}\wp_{k,t+1} + \wp_{k,t+2^{n+2}+2}. \tag{23}$$

Proof. From the Lemma 2.1, we have

$$2^{(2^{n+1}+1)} + k\eta_1 2^{(2^{n+1})} + \eta_1^{2(2^{n+1}+1)} = \wp_{k,2^{n+1}}\eta_1^{2(2^n+1)} \tag{24}$$

$$2^{(2^{n+1}+1)} + k\eta_2 2^{(2^{n+1})} + \eta_2^{2(2^{n+1}+1)} = \wp_{k,2^{n+1}}\eta_2^{2(2^n+1)}. \tag{25}$$

Now, by multiplying Equation (24) by $\dfrac{\eta_1^t}{\eta_1 - \eta_2}$ and Equation (25) by $\dfrac{\eta_2^t}{\eta_1 - \eta_2}$ and subtracting, we obtain

$$2^{(2^{n+1}+1)}\Big(\frac{\eta_1^t - \eta_2^t}{\eta_1 - \eta_2}\Big) + k2^{(2^{n+1})}\Big(\frac{\eta_1^{t+1} - \eta_2^{t+1}}{\eta_1 - \eta_2}\Big) + \Big(\frac{\eta_1^{t+2^{n+2}+2} - \eta_2^{t+2^{n+2}+2}}{\eta_1 - \eta_2}\Big)$$

$$= \wp_{k,2^{n+1}}\Big(\frac{\eta_1^{t+2^{n+1}+2} - \eta_2^{t+2^{n+1}+2}}{\eta_1 - \eta_2}\Big)$$

That is

$$\Im_{k,t+2^{n+1}+2}\wp_{k,2^{n+1}} = 2^{(2^{n+1}+1)}\Im_{k,t} + k2^{(2^{n+1})}\Im_{k,t+1} + \Im_{k,t+2^{n+2}+2}.$$

Furthermore, by multiplying Equation (24) by η_1^t and Equation (25) by η_2^t and adding, we obtain the desired result

$$\wp_{k,t+2^{n+1}+2}\wp_{k,2^{n+1}} = 2^{(2^{n+1}+1)}\wp_{k,t} + k2^{(2^{n+1})}\wp_{k,t+1} + \wp_{k,t+2^{n+2}+2}.$$ □

Theorem 2.2. *Let* $n, t \in \mathbb{Z}^+$ *with* $t \geq 1$. *Then*

$$(1) \quad k^n \Im_{k,n+t} 2^{(2^{r+1}n)} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{(2^{r+1})s} \wp^i_{k,2^{r+1}} \Im_{k,2^{r+1}(i+2j)+2(i+j)+t}, \tag{26}$$

$$(2) \quad k^n \wp_{k,n+t} 2^{(2^{r+1}n)} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{(2^{r+1})s} \wp^i_{k,2^{r+1}} \wp_{k,2^{r+1}(i+2j)+2(i+j)+t}. \tag{27}$$

Proof. From the Lemma 2.1, we have

$$k 2^{2^{r+1}} \eta_1 = \wp_{k,2^{r+1}} \eta_1^{2(2^r+1)} - 2^{2^{r+1}+1} - \eta_1^{2(2^{r+1}+1)},$$

$$k 2^{2^{r+1}} \eta_2 = \wp_{k,2^{r+1}} \eta_2^{2(2^r+1)} - 2^{2^{r+1}+1} - \eta_2^{2(2^{r+1}+1)}.$$

Thanks to the Multinomial Theorem, by applying it, we obtain

$$k^n 2^{2^{r+1}n} \eta_1^n = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{2^{r+1}s} \wp^i_{k,2^{r+1}} \eta_1^{(2^{r+1}+2)i} \eta_1^{(2^{r+2}+2)j}, \tag{28}$$

$$k^n 2^{2^{r+1}n} \eta_2^n = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{2^{r+1}s} \wp^i_{k,2^{r+1}} \eta_2^{(2^{r+1}+2)i} \eta_2^{(2^{r+2}+2)j}. \tag{29}$$

Now, multiplying Equation (28) by $\frac{\eta_1^t}{\eta_1 - \eta_2}$ and Equation (29) by $\frac{\eta_2^t}{\eta_1 - \eta_2}$ and subtracting, we get

$$k^n \Im_{k,n+t} 2^{(2^{r+1}n)} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{(2^{r+1})s} \wp^i_{k,2^{r+1}} \Im_{k,2^{r+1}(i+2j)+2(i+j)+t}.$$

Furthermore, multiplying Equation (28) by η_1^t and Equation (29) by η_2^t and adding, we obtain

$$k^n \wp_{k,n+t} 2^{(2^{r+1}n)} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{j+s} 2^{(2^{r+1})s} \wp^i_{k,2^{r+1}} \wp_{k,2^{r+1}(i+2j)+2(i+j)+t}. \qquad \square$$

Theorem 2.3. *Let* $n, t \in \mathbb{Z}^+$ *with* $t \geq 1$. *Then*

$$(1) \quad \Im_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp^i_{k,2^{r+1}} \Im_{k,(2^{r+1}+2)i+j+t}, \tag{30}$$

$$(2) \quad \wp_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp^i_{k,2^{r+1}} \wp_{k,(2^{r+1}+2)i+j+t}. \tag{31}$$

Proof. Using the Lemma 2.1, we have

$$\eta_1^{2(2^{r+1}+1)} = \wp_{k,2^{r+1}} \eta_1^{2(2^r+1)} - 2^{2^{r+1}+1} - k 2^{2^{r+1}} \eta_1,$$

$$\eta_1^{2(2^{r+1}+1)} = \wp_{k,2^{r+1}} \eta_2^{2(2^r+1)} - 2^{2^{r+1}+1} - k 2^{2^{r+1}} \eta_2.$$

By employing the multinomial theorem, we obtain

$$\eta_1^{2(2^{r+1}+1)n} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp^i_{k,2^{r+1}} \eta_1^{(2^{r+1}+2)i} \eta_1^j, \tag{32}$$

$$\eta_2^{2(2^{r+1}+1)n} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp^i_{k,2^{r+1}} \eta_2^{(2^{r+1}+2)i} \eta_2^j. \tag{33}$$

First, by multiplying Equation (32) by $\frac{\eta_1^t}{\eta_1-\eta_2}$ and Equation (33) by $\frac{\eta_2^t}{\eta_1-\eta_2}$ and subtracting, we attain

$$\Im_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp_{k,2^{r+1}}^i \Im_{k,(2^{r+1}+2)i+j+t}.$$

Moreover, multiplying Equation (32) by η_1^t and Equation (33) by η_2^t and adding, we obtain

$$\wp_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} 2^{2^{r+1}(j+s)+s} \wp_{k,2^{r+1}}^i \wp_{k,(2^{r+1}+2)i+j+t}. \qquad \square$$

Theorem 2.4. *Let $n, t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

$$(1) \quad \Im_{k,(2^{r+1}+2)n+t} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \Im_{k,(2^{r+2}+2)i+j+t}, \tag{34}$$

$$(2) \quad \wp_{k,(2^{r+1}+2)n+t} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \wp_{k,(2^{r+2}+2)i+j+t}. \tag{35}$$

Proof. From the Lemma 2.1, we have

$$\begin{aligned} \wp_{k,2^{r+1}} \eta_1^{2(2^r+1)} &= 2^{2^{r+1}+1} + k 2^{2^{r+1}} \eta_1 + \eta_1^{2(2^{r+1}+1)}, \\ \wp_{k,2^{r+1}} \eta_2^{2(2^r+1)} &= 2^{2^{r+1}+1} + k 2^{2^{r+1}} \eta_2 + \eta_2^{2(2^{r+1}+1)}. \end{aligned}$$

By using the multinomial theorem, we get

$$\eta_1^{2(2^r+1)n} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \eta_1^{2(2^{r+1}+1)i} \eta_1^j, \tag{36}$$

$$\eta_2^{2(2^r+1)n} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \eta_2^{2(2^{r+1}+1)i} \eta_2^j. \tag{37}$$

Now, by multiplying Equation (36) by $\frac{\eta_1^t}{\eta_1-\eta_2}$ and Equation (37) by $\frac{\eta_2^t}{\eta_1-\eta_2}$ and subtracting, we obtain

$$\Im_{k,(2^{r+1}+2)n+t} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \Im_{k,(2^{r+2}+2)i+j+t}.$$

Again, multiplying Equation (36) by η_1^t and Equation (37) by η_2^t and adding, we achieve

$$\wp_{k,(2^{r+1}+2)n+t} \wp_{k,2^{r+1}}^n = \sum_{i+j+s=n} \binom{n}{i,j} k^j 2^{2^{r+1}(j+s)+s} \wp_{k,(2^{r+2}+2)i+j+t}. \qquad \square$$

The following Lemmas 2.2 and 2.3 play a significant role in proving the Theorems 2.5 and 2.6.

Lemma 2.2. *Let $t \in \mathbb{Z}^+$ with $t \geq 1$. Then prove that*

$$(1) \quad \eta_1^{2t} = \frac{\Im_{k,2t}}{k} \eta_1 \sqrt{\delta} - 2 \frac{\wp_{k,2t-1}}{k}, \tag{38}$$

$$(2) \quad \eta_2^{2t} = -\frac{\Im_{k,2t}}{k} \eta_2 \sqrt{\delta} - 2 \frac{\wp_{k,2t-1}}{k}. \tag{39}$$

Proof. Let us consider

$$\begin{aligned}
\frac{\Im_{k,2t}}{k}\eta_1\sqrt{\delta}-2\frac{\wp_{k,2t-1}}{k} &= \frac{1}{k}\big(\frac{\eta_1^{2t}-\eta_2^{2t}}{\eta_1-\eta_2}\big)\eta_1(\eta_1-\eta_2)-2\big(\frac{\eta_1^{2t-1}+\eta_2^{2t-1}}{k}\big)\\
&= \frac{\eta_1^{2t+1}+2\eta_2^{2t-1}-2\eta_1^{2t-1}-2\eta_2^{2t-1}}{k}\\
&= \frac{\eta_1^{2t-1}(\eta_1^2-2)}{k}\\
&= \frac{\eta_1^{2t-1}(k\eta_1+2-2)}{k}\\
&= \eta_1^{2t}
\end{aligned}$$

This completes the proof of (1).
The proof of result (2) is similar to the result (1), hence we omit the proof. □

Lemma 2.3. *Let* $t\in\mathbb{Z}^+$ *with* $t\geq 1$. *Then show that*

$$(1)\quad \eta_1^{2t+1}=\frac{\wp_{k,2t+1}}{k}\eta_1-2\frac{\Im_{k,2t}}{k}\sqrt{\delta}, \tag{40}$$

$$(2)\quad \eta_2^{2t+1}=\frac{\wp_{k,2t+1}}{k}\eta_2+2\frac{\Im_{k,2t}}{k}\sqrt{\delta}. \tag{41}$$

Proof. Consider

$$\begin{aligned}
\frac{\wp_{k,2t+1}}{k}\eta_1-2\frac{\Im_{k,2t}}{k}\sqrt{\delta} &= \frac{1}{k}\Big\{(\eta_1^{2t+1}+\eta_2^{2t+1})\eta_1-2\big(\frac{\eta_1^{2t}-\eta_2^{2t}}{\eta_1-\eta_2}\big)(\eta_1-\eta_2)\Big\}\\
&= \frac{1}{k}\big(\eta_1^{2t+2}-2\eta_2^{2t}-2\eta_1^{2t}+2\eta_2^{2t}\big)\\
&= \frac{1}{k}\big(\eta_1^{2t+2}-2\eta_1^{2t}\big)\\
&= \frac{\eta_1^{2t}}{k}\big(\eta_1^2-2\big)\\
&= \frac{\eta_1^{2t}}{k}\big(k\eta_1+2-2\big)\\
&= \eta_1^{2t+1}.
\end{aligned}$$

Thus the result (1).
The proof of (2) is analogous to (1), thus we omit the proof. □

Theorem 2.5. *Let* $t\in\mathbb{Z}^+$ *with* $t\geq 1$. *Then*

$$(1)\quad k\Im_{k,s+2t}+2\wp_{k,2t-1}\Im_{k,s}=\Im_{k,2t}\wp_{k,s+1}, \tag{42}$$

$$(2)\quad k\wp_{k,s+2t}+2\wp_{k,2t-1}\wp_{k,s}=\delta\Im_{k,2t}\Im_{k,s+1}, \tag{43}$$

$$(3)\quad k\Im_{k,s+2t+1}=\wp_{k,2t+1}\Im_{k,s+1}-2\Im_{k,2t}\wp_{k,s}\sqrt{\delta}, \tag{44}$$

$$(4)\quad k\wp_{k,s+2t+1}=\wp_{k,2t+1}\wp_{k,s+1}-2\delta\Im_{k,2t}\Im_{k,s}. \tag{45}$$

Proof. From the Lemma 2.2, we have

$$\eta_1^{2t+s} = \frac{\Im_{k,2t}}{k}\eta_1^{s+1}\sqrt{\delta} - 2\frac{\wp_{k,2t-1}}{k}\eta_1^{s}, \tag{46}$$

$$\eta_2^{2t+s} = -\frac{\Im_{k,2t}}{k}\eta_2^{s+1}\sqrt{\delta} - 2\frac{\wp_{k,2t-1}}{k}\eta_2^{s}. \tag{47}$$

By multiplying Equation (46) by $\frac{1}{\eta_1-\eta_2}$ and Equation (47) by $\frac{1}{\eta_1-\eta_2}$ and subtracting, we obtain

$$k\Im_{k,s+2t} + 2\wp_{k,2t-1}\Im_{k,s} = \Im_{k,2t}\wp_{k,s+1}.$$

Lastly, adding the Equations (46) and (47), we get

$$k\wp_{k,s+2t} + 2\wp_{k,2t-1}\wp_{k,s} = \delta\Im_{k,2t}\Im_{k,s+1}.$$

By applying the Lemma 2.3, we have

$$\eta_1^{2t+s+1} = \frac{\wp_{k,2t+1}}{k}\eta_1^{s+1} - 2\sqrt{\delta}\frac{\Im_{k,2t}}{k}\eta_1^{s}, \tag{48}$$

$$\eta_2^{2t+s+1} = \frac{\wp_{k,2t+1}}{k}\eta_2^{s+1} + 2\sqrt{\delta}\frac{\Im_{k,2t}}{k}\eta_2^{s} \tag{49}$$

Now, by multiplying Equation (48) by $\frac{1}{\eta_1-\eta_2}$ and Equation (49) by $\frac{1}{\eta_1-\eta_2}$ and subtracting, we obtain

$$k\Im_{k,s+2t+1} = \wp_{k,2t+1}\Im_{k,s+1} - 2\Im_{k,2t}\wp_{k,s}\sqrt{\delta}.$$

Furthermore, adding the Equations (48) and (49), we get

$$k\wp_{k,s+2t+1} = \wp_{k,2t+1}\wp_{k,s+1} - 2\delta\Im_{k,2t}\Im_{k,s}. \qquad \square$$

Theorem 2.6. *Let $n, s, t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

$$(1)\ \sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\Im_{k,2ti+s} = \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\Im_{k,n+s}, & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n-1}{2}}\wp_{k,n+s}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

$$(2)\ \sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\wp_{k,2ti+s} = \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\wp_{k,n+s}, & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n+1}{2}}\Im_{k,n+s}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

$$(3)\ \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}k^{-i}\wp_{k,2t+1}^{i}\Im_{k,2t(n-i)+n} = \begin{cases} 0, & \text{if } n \text{ is an even integer;} \\ (-2)^{n+1}(k)^{-n}\Im_{k,2t}^{n}\delta^{(\frac{n-1}{2})}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

$$(4)\ \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}k^{-i}\wp_{k,2t+1}^{i}\wp_{k,2t(n-i)+n} = \begin{cases} (2)^{n+1}(k)^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}, & \text{when } n \text{ is an even integer;} \\ 0, & \text{when } n \text{ is an odd integer.} \end{cases}$$

Proof. Applying the Lemma 2.2, we have

$$\eta_1^{2t} + 2\frac{\wp_{k,2t-1}}{k} = \frac{\Im_{k,2t}}{k}\eta_1\sqrt{\delta},$$
$$\eta_2^{2t} + 2\frac{\wp_{k,2t-1}}{k} = -\frac{\Im_{k,2t}}{k}\eta_2\sqrt{\delta}.$$

Thanks to the binomial theorem. By using it, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}k^{i-n}2^{n-i}\wp_{k,2t-1}^{(n-i)}(\eta_1^{2ti}) = k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\eta_1^{n}, \tag{50}$$
$$\sum_{i=0}^{n}\binom{n}{i}k^{i-n}2^{n-i}\wp_{k,2t-1}^{(n-i)}(\eta_2^{2ti}) = (-1)^n k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\eta_2^{n}. \tag{51}$$

Now, by multiplying Equation (50) by $\frac{\eta_1^s}{\eta_1-\eta_2}$ and Equation (51) by $\frac{\eta_2^s}{\eta_1-\eta_2}$ and subtracting, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\left(\frac{\eta_1^{2ti+s}-\eta_2^{2ti+s}}{\eta_1-\eta_2}\right)$$
$$= \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\left(\frac{\eta_1^{n+s}-\eta_2^{n+s}}{\eta_1-\eta_2}\right), & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n-1}{2}}\left(\eta_1^{n+s}+\eta_2^{n+s}\right), & \text{if } n \text{ is an odd integer,} \end{cases}$$

which implies

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\Im_{k,2ti+s} = \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\Im_{k,n+s}, & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n-1}{2}}\wp_{k,n+s}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

This completes the proof of (1).

Again, by multiplying Equation (50) by η_1^s and Equation (51) by η_2^s and adding, we get

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\left(\eta_1^{2ti+s}+\eta_2^{2ti+s}\right)$$
$$= \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\left(\eta_1^{n+s}+\eta_2^{n+s}\right), & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n+1}{2}}\left(\frac{\eta_1^{n+s}-\eta_2^{n+s}}{\eta_1-\eta_2}\right), & \text{if } n \text{ is an odd integer,} \end{cases}$$

which gives

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}2^{n-i}\wp_{k,2t-1}^{(n-i)}\wp_{k,2ti+s} = \begin{cases} k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}\wp_{k,n+s}, & \text{if } n \text{ is an even integer;} \\ k^{-n}\Im_{k,2t}^{n}\delta^{\frac{n+1}{2}}\Im_{k,n+s}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

Thus the result (2).

From the Lemma 2.3, we have

$$\eta_1^{2t+1} - \frac{\wp_{k,2t+1}}{k}\eta_1 = -2\frac{\Im_{k,2t}}{k}\sqrt{\delta},$$
$$\eta_2^{2t+1} - \frac{\wp_{k,2t+1}}{k}\eta_2 = 2\frac{\Im_{k,2t}}{k}\sqrt{\delta}.$$

Applying the binomial theorem, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i k^{-i}\wp_{k,2t+1}^{i}(\eta_1^{2t(n-i)+n}) = (-1)^n k^{-n}2^n\Im_{k,2t}^{n}\delta^{\frac{n}{2}}, \tag{52}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i k^{-i}\wp_{k,2t+1}^{i}(\eta_2^{2t(n-i)+n}) = k^{-n}2^n\Im_{k,2t}^{n}\delta^{\frac{n}{2}}. \tag{53}$$

By multiplying Equation (52) by $\dfrac{1}{\eta_1-\eta_2}$ and Equation (53) by $\dfrac{1}{\eta_1-\eta_2}$ and subtracting, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i k^{-i}\wp_{k,2t+1}^{i}\Im_{k,2t(n-i)+n} = \begin{cases} 0, & \text{if } n \text{ is an even integer;} \\ (-2)^{n+1}(k)^{-n}\Im_{k,2t}^{n}\delta^{(\frac{n-1}{2})}, & \text{if } n \text{ is an odd integer,} \end{cases}$$

Thus the proof of (3).
Finally, by adding equations (52) and (53), we get

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i k^{-i}\wp_{k,2t+1}^{i}\wp_{k,2t(n-i)+n} = \begin{cases} (2)^{n+1}(k)^{-n}\Im_{k,2t}^{n}\delta^{\frac{n}{2}}, & \text{when } n \text{ is an even integer;} \\ 0, & \text{when } n \text{ is an odd integer.} \end{cases}$$

Hence the result (4). □

3 Conclusions

In this paper, we established a few crucial identities consisting of k-Jacobsthal and k-Jacobsthal–Lucas numbers. In addition, we obtained different binomial sums of k-Jacobsthal and k-Jacobsthal–Lucas numbers by using multinomial theorem. These results are original and the idea of the proof is inventive and distinct.

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